Hamiltonian Paths in $L$–shaped Grid Graphs

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Abstract

Grid graphs are subgraphs of the infinite 2-dimensional integer grid. The Hamiltonian path problem for general grid graphs is a well-known NP-complete problem. In this paper, we present necessary and sufficient conditions for the existence of a Hamiltonian path between two given vertices in $L$–shaped grid graphs. We also show that a Hamiltonian path between two given vertices of a $L$–shaped grid graph can be computed in linear time.

Keywords: Grid graph, Hamiltonian path, $L$–shaped grid graph, NP-complete.

1. Introduction

Hamiltonian path in a graph is a simple path that visits every vertex exactly once. The problem of deciding whether a given graph has a Hamiltonian path is a well-known NP-complete problem and has many applications [5]. However, for some special classes of graphs polynomial-time algorithms have been found. For more related results on Hamiltonian paths in general graphs see [3, 8, 19].

A grid graph is a graph in which vertices lie on integer coordinates and edges connect vertices that are separated by a distance of one. A solid grid graph is a grid graph without holes. A rectangular grid graph $R(m, n)$ is the subgraph of $G^\infty$ (the infinite grid graph) induced by $V(R) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$, where $v_x$ and $v_y$ are $x$ and $y$ coordinates of $v$ respectively. A $L$–shaped grid graph $L(m, n, k, l)$ is a grid graph obtained from a rectangular grid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper right (or bottom left) corner. Grid graphs can be useful representations in many applications such as robotic planning problems [6] and pathogen biology [21]. Myers [18] suggests modeling city blocks in which street intersection are vertices and streets are edges. He studied enumeration of tours in Hamiltonian rectangular lattice graphs. Luccio and Mugnia [17] suggest using a grid graph to represent a two-dimensional array type memory accessed by a read/write head moving up, down or across. The vertices correspond to the center of each cell and edges connect adjacent cells. Finding a path in the grid corresponds to accessing all the data. They studied Hamiltonian paths on rectangular chessboards.

As other applications we can mention the following. In the problem of embedding a graph in a given grid [4], the first step is to recognize if there are enough rooms in the host grid for the guest graph. If the guest graph is a path, then the problem makes relation to the well-known longest path and Hamiltonian path problems. If we would like to see if a given solid grid graph has a Hamiltonian path we may reach to the problem of finding a Hamiltonian path between two given vertices.
In the offline exploration problem [10], a mobile robot with limited sensor should visit every cell in a known cellular room without obstacles in order to explore it and return to start point such that the number of multiple cell visits is small. In this problem, let the vertices correspond to the center of each cell and edges connect adjacent cells, then we have a grid graph with a given start and end points. Finding a Hamiltonian cycle in the grid graph corresponds to visiting each cell exactly once (i.e., a cycle containing all the vertices of the grid graph).

In the picturesque maze generation problem [9], we are given a rectangular black-and-white raster image and want to randomly generate a maze in which the solution path fills up the black pixels. The solution path is a Hamiltonian path of a subgraph induced by the vertices that correspond to the black cells.

Itai et al. [12] have shown that the Hamiltonian path problem for general grid graphs, with or without specified endpoints, is NP-complete. The problem for rectangular grid graphs, however, is in P requiring only linear-time. Later, Chen et al. [2] improved the algorithm of [12] and presented a parallel algorithm for the problem in mesh architecture. Lenhart and Umans [16] have presented a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs. Zamfirescu and Zamfirescu [22] have given sufficient conditions for a grid graph to be Hamiltonian and it is proved that all finite grid graphs of positive width have Hamiltonian line graphs.

Recently Hamiltonian cycle (path) problem in grid graphs have received much attention. Salman [20] introduced a family of grid graphs, i.e., alphabet grid graphs, and determined classes of alphabet grid graphs that contain Hamiltonian cycles. Islam et al. [11] showed that the Hamiltonian cycle problem in hexagonal grid graphs is NP-complete. Gordon et al. [7] proved that all connected, and locally connected triangular grid graphs are Hamiltonian, and gave a sufficient condition for a connected graph to be fully cycle extendable and also showed that the Hamiltonian cycle problem for triangular grid graphs is NP-complete. In [13], the authors proposed a linear-time algorithm for the Hamiltonian path problem for some small classes of grid graphs, namely $L$–alphabet, $C$–alphabet, $E$–alphabet, and $F$–alphabet grid graphs. $L$–alphabet grid graphs considered in [13] is a special case of $L$–shaped grid graphs. Some other results about grid graphs are investigated in [1, 14, 15, 23].

In this paper, we obtain necessary and sufficient conditions for the existence of a Hamiltonian path in $L$–shaped grid graphs, which are a special type of solid grid graphs, between two given vertices. Also, we show that a Hamiltonian path in this graph can be computed in linear time. This research can be considered as the first attempts to solve the problem in solid grid graphs.

The paper is organized as follows. In Section 1, some preliminary definitions, notations, and previous results are presented. Necessary conditions for the existence of a Hamiltonian path in $L$–shaped grid graphs are given in Section 3. In Section 4, we show how to obtain a Hamiltonian path for $L$–shaped grid graphs (sufficient conditions). The conclusion and future work are given in Section 5.

2. Preliminary definitions and previous results

In this section, we give a few definitions and introduce the corresponding notations. We then gather some previously established results on the Hamiltonian path problem in grid graphs which have been presented in [2, 12].

The two-dimensional integer grid $G^\infty$ is an infinite graph with vertex set of all the points of the Euclidean plane with integer coordinates. In this graph, there is an edge between any two vertices of unit distance. For a vertex $v$ of this graph, let $v_x$ and $v_y$ denote $x$ and $y$ coordinates of its corresponding point, respectively (sometimes we use $(v_x, v_y)$ instead of $v$). We color the vertices of the two-dimensional integer grid as black and white. A vertex $v$ is colored white if $v_x + v_y$ is even, and it is colored black otherwise.

A grid graph $G_v$ is a finite vertex-induced subgraph of the two-dimensional integer grid $G^\infty$. In a grid graph $G_v$, each vertex has degree at most four. Clearly, there is no edge between any two vertices of the same color. Therefore, $G_v$ is a bipartite graph. Note that any cycle or path in a bipartite graph alternates between black and white vertices. Suppose that $G = (V(G), E(G))$ is a graph with vertex set $V(G)$ and edge set $E(G)$. Assume $v \in V(G)$. The number of edges incident at $v$ in $G$ is called degree of the vertex $v$ in $G$ and is denoted by $\text{degree}(v)$.

A rectangular grid graph, denoted by $R(m, n)$ (or $R$ for short), is a grid graph whose vertex set is $V(R) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$. The size of $R(m, n)$ is defined to be $m \times n$. $R(m, n)$ is called odd-sized if $m \times n$ is odd, and it is called even-sized otherwise (see Fig. 1(a)). $R(m, n)$ is called a $k$–rectangle if $k = m \times n$.

A $L$–shaped grid graph, denoted by $L(m, n, k, l)$ (or $L$ for short), is a grid graph obtained from a rectangular grid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper right (or bottom left) corner, where $k, l \geq 1$ and $m, n > 1$. $L(m, n, k, l)$ is called even-sized if $m \times n - k \times l$ is even, and it is called odd-sized otherwise.
In this paper without loss of generality, we consider \( L(m, n, k, l) \) where subgraph \( R(k, l) \) has been removed from the upper right corner of \( R(m, n) \). For \( m = 8, n = 5, k = 4, \) and \( l = 2 \) it is illustrated in Fig. 1(b). In the figures, we assume that \((1, 1)\) is the coordinates of the vertex in the upper left corner, except we explicitly change this assumption. Also in this paper, we will only consider the following three cases, other cases are isomorphic to these cases.

1. \( R(m, n) \) is even×even and \( R(k, l) \) is even×even, even×odd, or odd×odd.

2. \( R(m, n) \) is even×odd and \( R(k, l) \) is even×even, odd×even, even×odd, or odd×odd.

3. \( R(m, n) \) is odd×odd and \( R(k, l) \) is even×even, even×odd, or odd×odd.

In the following, we use \((G, s, t)\) to denote a grid graph \( G \) with two specified distinct vertices \( s \) and \( t \). Without loss of generality, we assume \( s_x \leq t_x \). \((G, s, t)\) is called Hamiltonian if there is a Hamiltonian path \( s \) and \( t \) in \( G \). In the following by Hamiltonian \((s, t)\)-path we mean a Hamiltonian path between \( s \) and \( t \).

**Definition 2.1.** Suppose that \( G(V_1 \cup V_2, E) \) is a bipartite graph such that \(|V_1| \geq |V_2|\) and the vertices of \( G \) colored by two colors, black and white. All the vertices of \( V_1 \) will be colored by one color, the majority color, and the vertices of \( V_2 \) by the minority color. The Hamilton path problem \( P(G, s, t) \) is color-compatible if

1. \( s \) and \( t \) have different colors and \( G \) is even-sized (\(|V_1| = |V_2|\)), or
2. \( s \) and \( t \) have the majority color and \( G \) is odd-sized (\(|V_1| = |V_2| + 1\))

**Definition 2.2.** Let \( G \) be a connected graph and \( V_1 \) be a subset of the vertex set \( V(G) \). \( V_1 \) is a vertex cut of \( G \) if \( G - V_1 \) is disconnected. A vertex \( v \) of \( G \) is a cut vertex of \( G \) if \( v \) is a vertex cut of \( G \). For an example, in Fig. 1(c) \{s, t\} is a vertex cut and in Fig. 1(d) \( t \) is a cut vertex.

An even-sized rectangular grid graph contains the same number of black and white vertices. Hence, the two end-vertices of any Hamiltonian path in the graph must have different colors. Similarly, in an odd-sized rectangular grid graph the number of vertices with the majority color is one more than the number of vertices with the minority color. Thus, the two end-vertices of any Hamiltonian path in such a graph must be the majority color. Hence, the color-compatibility of \( s \) and \( t \) is a necessary condition for a rectangular grid graph to be Hamiltonian. Furthermore, Itai et al. [12] showed that if one of the following conditions holds, then \( R(m, n) \) is not Hamiltonian:

1. \( s \) or \( t \) is a cut vertex or \{\( s, t \)\} is a vertex cut (Fig. 1(c) and 1(d)). Notice that, here, \( s \) or \( t \) is a cut vertex if \( R(m, n) \) is a 1-rectangle and either \( s \) or \( t \) is not a corner vertex, and \{\( s, t \)\} is a vertex cut if \( R(m, n) \) is a 2-rectangle and \[(2 \leq s_x = t_x \leq m - 1 \text{ and } n = 2) \text{ or } (2 \leq s_x = t_x \leq n - 1 \text{ and } m = 2)\].

2. All the cases that are isomorphic to the following cases:
   1. \( m \) is even, \( n \) is 3,
   2. \( s \) is black, \( t \) is white,
   3. \( s_y = 2 \text{ and } s_x < t_x \) (Fig. 1(e)) or \( s_y \neq 2 \) and \( s_x < t_x - 1 \) (Fig. 1(f)).

**Lemma 2.1.** Let \( G \) be a grid graph with two vertices \( s \) and \( t \). If \((G, s, t)\) satisfies condition (F1), then \((G, s, t)\) has no Hamiltonian \((s, t)\)-path.
Let $G$ be any grid graph. Let $s$ and $t$ be two given vertices of $G$ such that $(G, s, t)$ is color-compatible. If we can partition $(G, s, t)$ into $n$ subgraphs $G_1, G_2, \ldots, G_n$ such that $s, t \in G_n$ and in $V(G_1) \cup G_2 \cup \ldots \cup G_n$ the number of white and black vertices are equal, then $(G_n, s, t)$ is color-compatible.

Proof. Since in $V(G_1) \cup G_2 \cup \ldots \cup G_n$ the number of white and black vertices are equal, it follows that $|G_1 + G_2 + \ldots + G_n| = \text{even}$, where $|G|$ denotes the number of vertices of $G$. Let $G$ be odd-sized (resp. even-sized), then by Definition 2.1, $s$ and $t$ have the same color (resp. have different colors). Since $G$ is odd-sized (resp. even-sized) and $|G_1 + G_2 + \ldots + G_n| = \text{even}$, we conclude that $G_n = G \setminus (G_1 + G_2 + \ldots + G_{n-1})$ is odd-sized (resp. even-sized). Moreover since $s, t \in G_n$, $s$ and $t$ have the same color (resp. have different colors), and $G_n$ is odd-sized (resp. even-sized), it is clear that $(G_n, s, t)$ is color-compatible.

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Figure 2. The three types of separations; (a) and (b) horizontal separations, (c) and (d) vertical separations, (e) $L$–shaped separation type I, (f) $L$–shaped separation type II, and (g) $L$–shaped separation type III, where dotted lines indicate the separations.

Proof. If $s$ or $t$ is a cut vertex of $G$ or $[s, t]$ is a vertex cut of $G$, then $G - s, G - t$, and $G - [s, t]$ is disconnected and has at least two components, $G_1$ and $G_2$. Hence any paths between $s$ and $t$ must pass two times through $s$ or $t$. Therefore, $(G, s, t)$ has no Hamiltonian path (see Fig. 1(c) and 1(d)).

Definition 2.3. [12] A rectangular Hamiltonian path problem $P(R(m, n), s, t)$ is acceptable if it is color-compatible and $(R(m, n), s, t)$ does not satisfy any of conditions (F1) and (F2).

The following theorem was proved by Itai et al. [12].

Theorem 2.2. There exists a Hamiltonian $(s, t)$–path in $R(m, n)$ if and only if $P(R(m, n), s, t)$ is acceptable.

The following lemma states a result about the Hamiltonicity of even-sized rectangular grid graphs.

Lemma 2.3. [2] $R(m, n)$ has a Hamiltonian cycle if and only if it is even-sized and $m, n > 1$.

Fig. 1(g) shows a Hamiltonian cycle for an even-sized rectangular grid graph, found by Lemma 2.3. Every Hamiltonian cycle found by this lemma contains all the boundary edges on the three sides of the rectangular grid graph. This shows that for an even-sized rectangular grid graph $R$, we can always find a Hamiltonian cycle such that it contains all the boundary edges, except of exactly one side of $R$ which contains an even number of vertices.

3. Necessary conditions

In this section, we give necessary conditions for the existence of a Hamiltonian $(s, t)$–path in $L(m, n, k, l)$. We begin with the following definition.

Definition 3.1. A separation of a $L$–shaped grid graph $L(m, n, k, l)$ is a partition of $L(m, n, k, l)$ into two (or three) disjoint grid subgraphs $G_1, G_2$, and $G_3$, i.e., $V(L(m, n, k, l)) = V(G_1) \cup V(G_2) \cup V(G_3)$, and $V(G_1) \cap V(G_2) \cap V(G_3) = \emptyset$. $G_1$, $G_2$, and $G_3$ may be rectangular or $L$–shaped grid graphs. We have three types of separations, vertical, horizontal and $L$–shaped separations are shown in Fig. 2.

From Definitions 2.1 and 3.1, we have the following lemma.

Lemma 3.1. Let $G$ be any grid graph. Let $s$ and $t$ be two given vertices of $G$ such that $(G, s, t)$ is color-compatible. If we can partition $(G, s, t)$ into $n$ subgraphs $G_1, G_2, \ldots, G_n$, such that $s, t \in G_n$ and in $V(G_1) \cup G_2 \cup \ldots \cup G_n$ the number of white and black vertices are equal, then $(G_n, s, t)$ is color-compatible.
Since \( L(m,n,k,l) \) is bipartite, colors of vertices of any path must alternate between black and white. Hence, the color-compatibility of \( s \) and \( t \) in \( L(m,n,k,l) \) is a necessary condition for \( (L(m,n,k,l), s, t) \) to be Hamiltonian. Moreover, in addition to condition (F1) (as shown in Fig. 3(a)-(c)) whenever one of the following conditions is satisfied then \( (L(m,n,k,l), s, t) \) has no Hamiltonian \((s, t)\text{--}path\).

(F3) \( w \in V(L(m,n,k,l)), \deg(w) = 1, t \neq w, \text{ and } s \neq w \) (Fig. 3(d)).

(F4) \( L(m,n,1,1) \) is even-sized, \( m-1 = even > 2, n-1 = even > 2, s = (m-1, 2), \) and \( t \neq (m-1, 1) \) or \( t \neq (m, 2) \) (here the role of \( s \) and \( t \) can be swapped; i.e., \( t = (m-1, 2) \) and \( s \neq (m-1, 1) \)) (Fig. 3(e)).

(F5) \( L(m,n,k,l) \) is odd-sized, \( n-l = 2, m-k = odd \geq 3, \) and
   (i) \( s_t, t_s > m-k \) (Fig. 3(f)); or
   (ii) \( s = (m-k, n) \) and \( t_s > m-k \) (Fig. 3(g)).

(F6) \( L(m,n,k,l) \) is even-sized, \( n-l = 2, m-k = 2, \) and
   (i) \( s = (1, n-l) \) and \( t_s > 2 \) (Fig. 4(a)); or
   (ii) \( s = (2, n) \) and \( t_s < l \) (here the role of \( s \) and \( t \) can be swapped; i.e., \( t = (2, n) \) and \( s_t \leq l \)) (Fig. 4(b)).

(F7) \( L(m,n,k,l) \) is even-sized and
   (i) \( n = 3, l = 1, m-k = even > 2, s = (m-k-1, 1), \text{ and } t = (m-k, 3) \) (Fig. 4(c)); or
   (ii) \( m = 3, k = 1, \text{ and } n-l = even > 2, s = (1, l+1), \text{ and } t = (m, l+2) \) (Fig. 4(d)).

(F8) \( L(m,n,k,l) \) is even-sized and \([(m-k = 2 \text{ and } n-l > 2) \text{ or } (n-l = 2 \text{ and } m-k > 2)] \). Let \( G_1, G_2 \) be a vertical (or horizontal) separation of \( L(m,n,k,l) \) such that \( G_1 \) is a 3-rectangle grid graph, \( G_2 \) is a 2-rectangle grid graph, and exactly two vertices \( u \) and \( v \) are in \( G_1 \) that are connected to \( G_2 \). Let \( s' = s \) and \( t' = t \), if \( s' \text{ or } t' \notin G_1 \text{ then } s' = u \text{ or } t' = u \). And \((G_1, s, t)\) satisfies condition (F2) (Fig. 4(e)-(g)).

(F9) \( L(m,n,k,l) \) is even-sized and \([(m-k = 3 \text{ and } n-l \geq 3) \text{ or } (m-k > 3 \text{ and } n-l = 3)] \). Let \( G_1, G_2 \) be a vertical (horizontal or \( L \)-shaped) separation of \( L(m,n,k,l) \) such that \( G_1 \) and \( G_2 \) are even-sized, \( G_1 \) is a 3-rectangle grid graph, and \( G_2 \) is
   (1) a rectangular grid graph (see Fig. 5(a) and 5(c)), or
   (2) a \( L \)-shaped grid graph, where \( m \times n = even \times odd, k \times l = odd \times even, n-l = 3, \text{ and } m-k \geq 5 \). Here, \( V(G_1) = \{m-k \leq x \leq m \text{ and } l+1 \leq y \leq n\} \text{ and } G_2 = L(m,n,k,l) \setminus G_1 \) (see Fig. 5(b) and 5(d)).
Let exactly three vertices $v$, $w$ and $u$ are in $G_1$ that are connected to $G_2$. Let $s' = s$ and $t' = t$, if $s'$ (or $t'$) $\notin G_1$ then $s = w$ (or $t = w$). And $(G_1, s', t)$ satisfies condition (F2).

**Definition 3.2.** A $L$-shaped Hamiltonian path problem $P(L(m, n, k, l), s, t)$ is acceptable if it is color compatible and $(L(m, n, k, l), s, t)$ does not satisfy any of conditions (F1) and (F3)-(F9).

**Definition 3.3.** The length of a path in a grid graph means the number of vertices of the path. In any grid graph, the length of any path between two same-colored vertices is odd and the length of any path between two different-colored vertices is even.

**Theorem 3.2.** Let $(L(m, n, k, l), s, t)$ be a $L$-shaped grid graph and $s$ and $t$ be two distinct vertices of it. If $(L(m, n, k, l), s, t)$ is Hamiltonian, then $(L(m, n, k, l), s, t)$ is acceptable.

**Proof.** We prove the contraposition that if $(L(m, n, k, l), s, t)$ is not acceptable, then $(L(m, n, k, l), s, t)$ has no Hamiltonian $(s, t)$-path. It is obvious that if $(L(m, n, k, l), s, t)$ is not color-compatible, then $(L(m, n, k, l), s, t)$ has no Hamiltonian $(s, t)$-path. Thus, without loss of generality, assume that $(L(m, n, k, l), s, t)$ is color-compatible. In the following, we will show that if one of the conditions (F1) and (F3)-(F9) holds, then $(L(m, n, k, l), s, t)$ has no Hamiltonian $(s, t)$-path.

For condition (F1), see the proof of the Lemma 2.1 (Fig. 3(a)-(c)).

If condition (F3) satisfies, then $\text{degree}(w) = 1$, so $w$ could not be in any path from $s$ to $t$, except when $s = w$ or $t = w$; see Fig. 3(d).

For condition (F4), consider Fig. 3(e). It is easy to see that there is no Hamiltonian $(s, t)$-path in $(L(m, n, 1, 1)$ containing both of the vertices $u$ and $v$, assuming $t \neq u$, $t \neq v$ (or $s \neq u$ where $t = (n - 1, 2)$).

For condition (F5), consider Fig. 3(f) and 3(g). Let $[G_1, G_2]$ be a vertical separation of $(L(m, n, k, l)$ such that $G_1 = R(m - k, n)$ and $G_2 = R(k, n - l)$. Since $n - l = 2$, it follows that $G_1$ is even-sized. Moreover, since $L(m, n, k, l)$ is odd-sized, we conclude that $G_1$ is odd-sized. Clearly, the Hamiltonian path $P$ must enter to $G_1$ through one of the vertices $u$ (or $v$) then the path $P$ leaves $G_1$ after visiting the vertices of $G_1$ by $v$ (or $u$). Since $G_1$ is odd-sized and $v$ and $u$ have different colors, $(G_1, v, u)$ is not acceptable. Thus, by Theorem 2.2 $(G_1, u, v)$ does not have any Hamiltonian $(v, u)$-path. Hence, $(L(m, n, k, l), s, t)$ has no Hamiltonian $(s, t)$-path.

For condition (F6), consider Fig. 4(a) and 4(b). We can easily see that there is no Hamiltonian $(s, t)$-path in $(L(m, n, k, l)$ containing both of the vertices $u$ and $v$.

For condition (F7), assume that $n = 3$. Consider Fig. 4(e). The Hamiltonian path $P$ which starts from $s$ must pass from $u$ for the first time, visits vertex $v$, and then passes through the remaining vertices and end at $t$. Clearly, one of the two vertices $w$ and $z$ is out of path. Hence, $(L(m, n, k, l)$ has no Hamiltonian $(s, t)$-path. For the case $m = 3$, the proof is the same as the case $n = 3$; see Fig. 4(d).

For conditions (F8) and (F9), assume that $(G_1, s', t)$ satisfies condition (F2). We show that there is no Hamiltonian $(s, t)$-path in $(L(m, n, k, l)$. Assume to the contrary that $(L(m, n, k, l)$ has a Hamiltonian path $P$.

For condition (F8), we have the following cases:

Case 1. $s, t \in G_1$. Consider Fig. 4(e). The Hamiltonian path $P$ of $(L(m, n, k, l)$ which starts from $s$ should pass through some vertices of $G_1$, leaves $G_1$ at $v$ (or $u$), then passes through all the vertices of $G_2$ and reenters to $G_1$ at $u$.
(or v) and passes through all the remaining vertices of G₁ and end at t. In this case by connecting v to u, we obtain a Hamiltonian path from s to t in G₁, which contradicts to the assumption that (G₁, s′, t′) satisfies (F2).

Case 2. s ∈ G₁ and t ∈ G₂ (or t ∈ G₁ and s ∈ G₂). Consider Fig. 4(f) and 4(g). The Hamiltonian path P of L(m, n, k, l) starts from s and passes through all the vertices of G₁ (or G₂), leaves G₁ at one of the vertices v or u (or leaves G₂), then enters to G₂ (or enters to G₁ at one of the vertices u or v) and should pass through all the vertices of G₂ (or G₁) and end at t. This case is not possible because we assumed that (G₁, s′, t′) satisfies (F2), in this case s′ = s and t′ = u (or s′ = u and t′ = t).

For condition (F9), the following cases are possible:

Case 1. t ∈ G₁ and s ∈ G₁. The following subcases are possible for the Hamiltonian path P.

Subcase 1.1. The Hamiltonian path P of L(m, n, k, l) that starts from s may enter to G₁ for the first time through one of the vertices v, w or u and passes through all the vertices of G₁ and end at t; see Fig. 5(a). This case is not possible because we assumed that (G₁, s′, t′) satisfies (F2) (i′ = t and s′ = w).

Subcase 1.2. The Hamiltonian path P of L(m, n, k, l) may enter to G₁, passes through some vertices of it, then leaves it and enters it again and passes through all the remaining vertices of it and finally ends at t. In this case, two sub-paths of P which are in G₁ are called P₁ and P₂, P₁ from v to u (v to w or u to w) and P₂ from w to t (u to t or v to t); see Fig. 5(b). This case is not also possible because the size of P₁ is odd (even) and the size of P₂ is even (odd), then |P₁ + P₂| is odd while G₁ is even, which is a contradiction.

Case 2. s, t ∈ G₁. The following subcases may be considered:

Subcase 2.1. The Hamiltonian path P of L(m, n, k, l) starts from s passes through some vertices of G₁, leaves G₁ at v (or u), then passes through all the vertices of G₂ and reenters to G₁ at w goes to u (or v) and passes through all the remaining vertices of G₁ and ends at t; Fig. 5(c). In this case by connecting v (or u) to w we obtain a Hamiltonian path from s to t in G₁, which contradicts to the assumption that (G₁, s′, t′) satisfies (F2).

Subcase 2.2. The Hamiltonian path P of L(m, n, k, l) starts from s leaves G₁ at v (or u), then passes through all the vertices of G₂ and reenters to G₁ at u (or v) goes to w and passes through all the remaining vertices of G₁ and ends at t; Fig. 5(d). In this case, two parts of P resides in G₁. The part P₁ starts from s ends at v (or u), and the part P₂ starts from u (or v) ends at t. The size of P₁ is even and the size of P₂ is odd while the size of G₁ is even, which is a contradiction.

Subcase 2.3. Another case that may imagine is that the Hamiltonian path P of L(m, n, k, l) starts from s leaves G₁ at w and reenters G₁ at v (or u) and then goes to t. But in this case vertex u (or v) cannot be in P, which is a contradiction. Thus, the proof of Theorem 3.2 is completed.

4. Sufficient conditions

In this section, we show that all acceptable L-shaped Hamiltonian path problems have solutions by showing how Hamiltonian paths can be made in each cases.

Theorem 4.1. The cases that are mentioned in Lemmas 4.2–4.5 include all possible cases that may occur in (L(m, n, k, l), s, t).

Before we prove Theorem 4.1, we establish four lemmas.

Definition 4.1. A separation is acceptable if all of its component are acceptable.

First, we solve the problem for the case m − k = 1 or n − l = 1.

Lemma 4.2. Suppose that P(L(m, n, k, l), s, t) is an acceptable Hamiltonian path problem with (m − k = 1, n − l > 1, and [(i > l and s = (1, l)) or (s_i > l and t = (1, l))], (n − l = 1, m − k > 1, s_i ≤ m − k, and t = (m, n)), or (m − k = 1, n − l = 1, s = (1, 1), and t = (m, n)). Then there is an acceptable separation for (L(m, n, k, l), s, t) and it has a Hamiltonian path.

Proof. We prove lemma for case m − k = 1 and n − l ≥ 1. Let (R₁, R₂) be a horizontal separation of L(m, n, k, l) such that R₁ = R(m − k, n′), where n′ = l, and R₂ = R(m, n − n′). Let s, p ∈ R₁, q, t ∈ R₂, q and p are adjacent, and p = (1, l). Consider the following cases for L(m, n, k, l). In two cases, we show that (R₁, s, p) and (R₂, q, t) are color-compatible and they are not in any of conditions (F1) and (F2). Also, we show that (L(m, n, k, l), s, t) has a Hamiltonian path.
1.1. Thus, (R1, s, p) and (R2, q, t) are color-compatible. Now, we show that (R1, s, p) and (R2, q, t) are not in conditions (F1) and (F2). Consider (R1, s, p). Since R1 is a 1-rectangle, it is enough to show that (R1, s, p) is not in condition (F1). Because of s = (1, 1) and p = (1, l), (R1, s, p) is not in condition (F1), and hence it is acceptable. Now, consider (R2, q, t). The condition (F1) holds, if

(i) n − l = 1 and t ≠ (m, n) or q ≠ (1, l + 1). Since q = (1, l + 1) and t = (m, n), thus this case is not possible.

(ii) n − l = 2 and 2 ≤ q5 = t5 ≤ m − 1. Since q5 = 1, we have t5 = s5 = 1 or t5 ≠ s5.

Thus (R2, q, t) is not in condition (F1). The condition (F2) occurs, when m = even and n − l = 3. Since q = (1, l + 1), a simple check shows that condition (F2) cannot occur. Hence, (R2, q, t) is acceptable. Now, we show that (L(m, n, k, l), s, t) has a Hamiltonian path. Since (R1, s, p) and (R2, q, t) are acceptable, by Theorem 2.2, (R1, s, t) and (R2, q, t) have a Hamiltonian path. Hence, we construct a Hamiltonian path in (R1, s, p) and (R2, q, t) by the algorithm in [2]; see Fig. 6(b). Then the Hamiltonian path for (L(m, n, k, l), s, t) can be obtained by connecting two vertices p and q as shown in Fig. 6(c). Notice that, here, if |R1| = 1 (resp. |R2| = 1), where |R| denotes the number of vertices of R, then s = p (resp. q = t); see Fig. 6(d).

Subcase 1.2. l = even. Since m − k = 1 and l = even, we conclude that R1 is odd×even and R2 is odd-sized. Moreover, since l = even and the vertex with coordinates (1, 1) is white, it is clear that p = (1, l) is black and q = (1, l + 1) is white. Thus (R1, s, p) and (R2, q, t) are color-compatible. (R1, s, p) is not in conditions (F1) and (F2), the proof is the same as Subcase 1.1. Hence, (R1, s, p) is acceptable. Consider (R2, q, t). Since R2 is odd-sized, it suffices to prove that (R2, q, t) is not in condition (F1). (R2, q, t) is not in condition (F1), the proof similar to Subcase 1.1. Thus, (R2, q, t) is acceptable. The Hamiltonian path in (L(m, n, k, l), s, t) is obtained similar to Subcase 1.1; see Fig. 6(e) and 6(f).

Case 2. L(m, n, k, l) is even-sized. In this case, s and t have different colors. By the same argument as in proof Case 1, the Hamiltonian path in (L(m, n, k, l), s, t) is obtained; see Fig. 6(g).

For case n − l = 1 and m − k > 1, let (1, 1) be the coordinates of the vertex in the bottom right corner in L(m, n, k, l), then this case is similar to case m − k = 1.

From now on, we assume m − k, n − l > 1.

Definition 4.2. Two nonincident edges (u1, v1) and (u2, v2) are parallel, if u1 (resp. v1) is adjacent to u2 and v1 (resp. u1) is adjacent to v2.

Lemma 4.3. Suppose that P(L(m, n, k, l), s, t) is an acceptable Hamiltonian path problem. Let L(m, n, k, l) be odd-sized. Then there is an acceptable separation for (L(m, n, k, l), s, t) and it has a Hamiltonian path.

Proof. The proof is by case analysis, thus we divide the proof into three cases. In all cases, first we are going to prove that (L(m, n, k, l), s, t) has an acceptable separation, then we show that it has a Hamiltonian path. Notice that, here, s and t are white. Let m × n = odd×odd. Since L(m, n, k, l) is odd-sized, we have k × l = even×even or even×odd. Hence, it is clear that m − k = odd and n − l = even if l = odd; otherwise n − l = odd. Now, let m × n = even×even or even×odd. Similarly since L(m, n, k, l) is odd-sized, we have k × l = odd×odd, and hence m − k = odd and n − l = odd if n = even; otherwise n − l = even. Moreover, Since m − k = odd ≥ 3, n − l > 1, and k, l ≥ 1, it follows that m ≥ 4 and n ≥ 3. Consider the following cases.
Case 1. $n = odd$ and $[(k > 1) \text{ or } (k = 1 \text{ and } n - l = 2)]$.

Subcase 1.1. $s, t, k \leq m - k$. Let $[R_1, R_2]$ be a vertical separation of $L(m, n, k, l)$ such that $R_1 = R(m', n)$, where $m' = m - k$, $R_2 = R(m' - n, m')$, and $s, t \in R_1$. Consider Fig. 7(a)-(c). Since $m - k = odd$ and $n = odd$, $R_1$ is oddodd. Thus, $(R_1, s, t)$ is color-compatible. Moreover, since $m - k \geq 3$ and $n \geq 3$, clearly $(R_1, s, t)$ is not in conditions (F1) and (F2), and hence $(R_1, s, t)$ is acceptable. Now, we show that $L(m, n, k, l)$ has a Hamiltonian path. Since $(R_1, s, t)$ is acceptable, by Theorem 2.2, it has a Hamiltonian $(s, t)$-path. So, we construct a Hamiltonian path in $(R_1, s, t)$ by the algorithm in [2]. Furthermore, since $R_2$ is even-sized, it has a Hamiltonian cycle by Lemma 2.3. Then by combining Hamiltonian cycle and path using two parallel edges $e_1$ and $e_2$ as shown in Fig. 7(d), a Hamiltonian $(s, t)$-path for $L(m, n, k, l)$ is obtained; see Fig. 7(e). Notice that since $n - l \geq 2$, there exists at least one edge for combining Hamiltonian cycle and path.

In the following, we describe combining a Hamiltonian path in $R_1$ with the constructed cycle in $R_2$. Any Hamiltonian path $P$ in $R_1$ contains all the vertices of $R_1$. Therefore, $P$ should contain a boundary edge of $R_1$ that has a parallel edge in $R_2$, except when $n - l = 2$, in this case $R_2$ may have no boundary edge parallel to any edge of $R_2$ (see Fig. 8(a)). But in this case, $t$ (or $s$) should be adjacent to $R_2$. Let an edge $(t, v)$ (or $(v, s)$) of $R_2$ be adjacent to $R_2$. Then $P$ must contain an edge $(t, u)$ (or $(u, s)$) such that $u$ is not adjacent to $R_2$, and $P$ can modify $P$ such that it contains a boundary edge adjacent to $R_2$ (see Fig. 8(c)). Using two parallel edges of $P$ and the Hamiltonian cycle of $R_2$ such as the two darkened edges of Fig. 8(c) we can combine them; see Fig. 8(d). Now, assume that $|R_2| = 2$. Consider Fig. 9(a). Let two vertices $v, u \in R_2$. Let $P$ be a Hamiltonian path in $R_1$. Then there exist exactly one edge $e_1$ such that $e_1 \in P$ is on boundary of $R_1$ facing $R_2$ (see Fig. 9(a)). Hence by merging $(v, u)$ to this edge, we obtain a Hamiltonian path for $(L(m, n, k, l), s, t)$ as shown in Fig. 9(b).

Subcase 1.2. $n - l = odd$ and $[(s, t, k \leq m - k, s, t > l, \text{ and } t > m - k)]$. Let $(1, 1)$ be the coordinates of the vertex in the bottom right corner in $L(m, n, k, l)$, then this case is similar to Subcase 1.1.

Subcase 1.3. $s, t, k \leq m - k$, and $[(n - l = odd \text{ and } s, t \leq l) \text{ or } (n - l = even)]$. This case is similar to Subcase 1.1, where $s, p \in R_1, q, t \in R_2, p$ and $q$ are adjacent, and

$$p = \begin{cases} (m - k, n); & \text{if } s \neq (m - k, n) \\ (m - k, n - 2); & \text{if } n - l = even > 2, s = (m - k, n), \text{ and } [(k > 3), (k = 2) \text{ and } t \neq (m - 2)], \text{ or} \\ (k = 3) \text{ and } t \neq (m - 1, n) \end{cases}$$
Figure 9. (a) and (b) combining a Hamiltonian path in $R_1$ with an edge $(u,v)$, (c) a horizontal separation of $L(m,n,k,l)$, (d) a Hamiltonian $(s,t)$–path in $L(m,n,k,l)$, (e) a vertical separation of $L(m,n,k,l)$, (f) a Hamiltonian $(s,t)$–path in $L(m,n,k,l)$.

Notice that, in this case, if $n - l = 2$ and $s = (m - k, n)$, then $(L(m,n,k,l), s,t)$ satisfies condition (F5), hence this case cannot occur. Consider Fig. 7(a)-(c). Clearly, $(R_1, s,p)$ and $(R_2, q,t)$ are color-compatible. $(R_1, s,p)$ is not in conditions (F1) and (F2), the proof is the same as Subcase 1.1. Hence, $(R_1, s,p)$ is acceptable. Now, consider $(R_2, q,t)$.

In this case, $R_2$ is even-sized or odd-sized (when $m = odd$) and is odd-sized (when $m = even$). The condition (F1) holds, if $(k = 2$ and $l + 2 = q_t = t_1 < n - 1$) or $(n - l = 2$ and $q_t = t_1 = m - k + 1$). A simple check shows that $(R_2, q,t)$ is not in condition (F1). The condition (F2) occurs, when

(i) $n - l = 3$ and $t$ is black. Since $t$ is white, this case does not occur.

(ii) $k = 3$ and $q_t < t_1 - 1$. The only case that occurs is $t = (m - 1, n)$. This is impossible, because $t \neq (m - 1, n)$.

Therefore, $(R_2, q,t)$ does not satisfy conditions (F1) and (F2), and hence $(R_2, q,t)$ is acceptable. The Hamiltonian path in $(L(m,n,k,l), s,t)$ is obtained similar to Subcase 1.1 of Lemma 4.2. Now, assume that $s = (m - k, n)$ and $[(k = 2$ and $t = (m - n - 2))$ or $(k = 3$ and $t = (m - 1, n)]$. Consider the following subcases.

Subcase 1.3.1. $k = 2$, $n - l = even > 2$, $s = (m - k, n)$, and $t = (m - 1, n)$. Notice that, in this case, $m = odd$. Let $[G_1,G_2]$ be a horizontal separation of $(L(m,n,k,l))$ such that $G_1 = L(m,n,k,l)$, where $n = n - 2$, $G_2 = R(m,n,n')$, $q,t \in G_1$, $s \in G_2$, $p \subseteq G_2$, and $q$ and $s$ are adjacent, and $p \subseteq (1, n' + 1)$. Consider Fig. 9(c). Since $n' = n - 2$, it follows that $n - n' = 2$. Thus, $G_2$ is odd-sized and $G_1$ is odd-sized. It is clear that $(G_1,q,t)$ and $(G_2,s,p)$ are color-compatible. Consider $(G_2,s,p)$. Since $m \geq 5$, it suffices to prove that $(G_2,s,p)$ is not in condition (F1). The condition (F1) holds, if $n - n' = 2$, $s \leq s_1 = t_1 < m - 1$. Since $p_2 = 1$, $(G_2,s,p)$ is not in condition (F1), and hence $(G_2,s,p)$ is acceptable. Now, consider $(G_1,q,t)$. Since $q_1 = 1$ and $t_1 > m - k$, it is obvious that $(G_1,q,t)$ is not in conditions (F1), (F3), and (F5), and hence $(G_1,q,t)$ is acceptable. In this case, $(G_1,q,t)$ is in Subcase 1.3. The Hamiltonian path in $(L(m,n,k,l), s,t)$ is obtained similar to Subcase 1.1 of Lemma 4.2 (as shown in Fig. 9(d)).

Subcase 1.3.2. $k = 3$, $n - l = even > 2$, $s = (m - k, n)$, and $t = (m - 1, n)$. Notice that, in this case, $m = even$. Let $[G_1,G_2]$ be a vertical separation of $(L(m,n,k,l))$ such that $G_1 = L(m,n',k,l)$, where $m' = m - 2$ and $k' = m' - (m - k)$, $G_2 = R(m,m',n,n')$, $s \subseteq G_1$, $t \subseteq G_2$, $p \subseteq G_2$, and $s$ and $t$ are adjacent, and $p \subseteq (m - 2,l + 1)$. Consider Fig. 9(e). It is clear that $(G_1,s,p)$ and $(G_2,q,t)$ are acceptable. In this case, $(G_1,s,p)$ is in Subcase 3.3.3, where $m' = 4$ or in Subcase 3.5.2, where $m' > 4$. The Hamiltonian path in $(L(m,n,k,l), s,t)$ is obtained similar to Subcase 1.1 of Lemma 4.2 (Fig. 9(f)).

Case 2. $n = odd$, $k > 1$, $n - l = even$, and $s_1,t_1 > m - k$. Notice that, in this case, $n - l > 4$. Moreover, since $m - k \geq 3$ and $k > 1$, it follows that $m \geq 5$.

Subcase 2.1. $n - l = 4$.

Subcase 2.1.1. $s_1,t_1 \leq n - 2$. Let $[G_1,G_2,G_3]$ be a $L$–shaped separation (type II) of $(L(m,n,k,l))$ such that $G_1 = L(m,n',k,l)$, where $n = n - 2$, $k = m - (m - k)$, and $m = t_k - 2$ if $m = odd$ and $t_k = m$; otherwise $m = t_k - 1$, $G_2 = R(m,m',n,n')$, and $G_3 = R(m,m,n)$, $s \subseteq G_1$ and $t \subseteq G_2$ such that $p \subseteq (1, n)$ and $q \subseteq (m,n)$ if $m = even$; otherwise $q = (m - 1,n')$. Let $w_1 \subseteq G_1$ such that $w_1 = w_2 + 3$. Since $n - l = 4$ and $n' = n - 2$, it follows that $n - n' = 2$ and $n' - l = 2$. Therefore, $G_2$ and $G_1$ are even-sized and $G_1$ is odd-sized. Clearly, $(G_1,s,p)$, $(G_2,q,t)$, and $(G_3,w,z)$ are color-compatible. By using the same argument as in proof Subcase 1.3.1, $(G_1,s,p)$ is not in conditions (F1), (F3), and (F5). Hence, $(G_1,s,p)$ is acceptable. Consider $(G_3,w,z)$. Since $m \geq 5$ and $m - n' = 2$, it suffices to prove that $(G_3,w,z)$ is not in condition (F1). Since $w_1 = 1$ and $w_2 = m - k$, it is clear that $(G_3,w,z)$ does not satisfy condition (F1). Thus, $(G_3,w,z)$ is acceptable. Now, consider $(G_2,q,t)$. In this case, $G_2$ is even-sized or odd-sized. The condition (F1) holds, if

(i) $m - m' = 1$. The only case that occurs is $t_k = m$. In this case, $|G_2| = 2$. We can easily see that $t = (m,n' - 1)$.
and \( q = (m, n') \).

(ii) \( q_t = t_x \neq m \), where \( m = \text{even} \), or \( q_t = t_x \neq m - 1 \), where \( m = \text{odd} \). A simple check shows that \( q_t = t_x = m \) or \( m - 1 \) or \( m - 1 \neq q_t \).

Therefore, \((G_2, q, t)\) is not in condition (F1). The condition (F2) occurs, when \( m - m' = 3 \). In this case, \( t = (m - 2, n - 2) \), where \( m = \text{odd} \), or \( t = (m - 2, n - 3) \), where \( m = \text{even} \), a simple check shows that \((G_2, q, t)\) does not satisfy condition (F2), and hence \((G_2, q, t)\) is acceptable. Now, we show that \((L(m, n, k, l))\) has a Hamiltonian path. In this case, \((G_1, s, p)\) is in Subcase 1.3. Since \((G_3, w, z), (G_2, q, t), \) and \((G_1, s, p)\) are acceptable, by Theorem 2.2 and Subcase 1.3, \((G_1, w, z)\), \((G_2, q, t)\), and \((G_1, s, p)\) have a Hamiltonian path, respectively. So, we construct a Hamiltonian path in \((G_3, w, z)\) and \((G_2, q, t)\) by the algorithm in [2] and in \((G_1, s, p)\) by Subcase 1.3. Finally, a Hamiltonian path for \((L(m, n, k, l), s, t)\) can be constructed by connecting vertices \( q \) and \( z \), and \( w \) and \( p \). The full construction of a Hamiltonian \((s, t)\)-path in \((L(m, n, k, l))\) is illustrated in Fig. 10(c) and 10(d).

Subcase 2.1.2. \( s_t > n - 2 \). This case is similar to Subcase 2.1.1, where \( G_3 = L(m, n', k, l), G_1 = R(m, n - n') \), \( G_2 = R(m - m', n - n'), n' = n - 2, m' - t_x - 2 \) if \( m = \text{even} \) and \( t_x = m \); otherwise \( m' - t_x - 1 \) (see Fig. 11(a) and 11(b)). In this case, \( p = (1, n' + 1) \) and \( q = (m, n' + 1) \) if \( m = \text{odd} \); otherwise \( q = (m - 1, n' + 1) \). By using the same argument as in proof Subcase 2.1.1, we derive \((G_1, s, p), (G_2, q, t), \) and \((G_3, w, z)\) are acceptable. Notice that, here, In this case, \((G_3, w, z)\) in Subcase 1.3.

Subcase 2.1.3. \( s_t \leq n - 2 \). This case is similar to Subcase 1.3.1. Notice that, here, if \( s_t \leq n - 2 \) and \( t_x > n - 2 \), the role of \( q \) and \( p \) can be swapped (i.e., \( s, p \in G_1 \) and \( q, t \in G_2 \)).

Subcase 2.2. \( m = \text{odd} \).

Subcase 2.2.1. \( s_t \leq l + 4 \). This case is similar to Subcase 1.3.1, where \( n' = l + 4 \) and \( s, t \in G_1 \). Since \( n - l = \text{even} \geq 4 \) and \( n' = l + 4 \), it follows that \( n - l = 4 \) and \( n - n' = \text{even} \geq 2 \). Therefore, \( G_2 \) is even-sized and \( G_1 \) is odd-sized. By Lemma 3.1, \((G_1, s, t)\) is color-compatible. Since \( m - k \geq \text{odd} \geq 3 \) and \( n - l = 4 \), it is clear that \((G_1, s, t)\) is not in conditions (F1), (F3), and (F5), and hence \((G_1, s, t)\) is acceptable. In this case, \((G_1, s, t)\) is in Subcase 2.1. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1. Since \( m \geq 5 \), thus there is at least one edge for combining Hamiltonian cycle and path.

Subcase 2.2.1. \( s_t > l + 4 \). This case is the same as Subcase 1.3.1, where \( n' = l + 3 \) and \( s, t \in G_2 \) (Fig. 11(c)). Since \( n' = l + 3 \) and \( l = \text{odd} \geq 1 \), we have \( n' = \text{even} \geq 4 \). Moreover, since \( n \geq 7 \) and \( n' = \text{even} \geq 4 \), it follows that
Figure 12. (a) and (b) A horizontal separation of \( L(m,n,k,l) \), (c)-(e) a Hamiltonian \((s,t)-\)path in \( L(m,n,k,l) \).

\( n - n' = \text{odd} \geq 3 \). Thus, \( G_2 \) is odd×odd and \( G_1 \) is even-sized. By Lemma 3.1, \((G_2,s,t)\) is color-compatible. Since \( m \geq 5 \) and \( n - n' \geq 3 \), it is obvious that \((G_2,s,t)\) is not in conditions (F1) and (F2), and hence \((G_2,s,t)\) is acceptable. The Hamiltonian path in \( (L(m,n,k,l),s,t) \) is obtained similar to Subcase 2.2.1.1; see Fig. 11(c). Notice that, the pattern for constructing a Hamiltonian cycle in \( G_1 \) is shown in Fig. 11(c).

Subcase 2.2.1.3. \( s_t > l + 4 \) and \( t_y \leq l + 4 \). This case is similar to Subcase 1.3.1, where \( n' = l + 4 \). As seen from the proof of Subcase 2.2.1.1, we know that \( n - n' = \text{even} \geq 2 \), \( G_2 \) is even-sized, and \( G_1 \) is odd-sized. In this case, if \( s_t \leq l + 4 \) and \( t_y > l + 4 \), the role of \( q \) and \( p \) can be swapped (i.e., \( s,p \in G_1 \) and \( q,t \in G_2 \)).

Subcase 2.2.2. \( m = \text{even} \).

Subcase 2.2.2.1. \( (s_t) = n - 3 \) and \( (t_y) = n - 3 \). This case is similar to Subcase 1.3.1, where \( n' = n - 3 \) and \( s,t \in G_1 \). Since \( n \geq 7 \) and \( n - l > 4 \), thus \( n' = \text{even} \geq 4 \) and \( n - n' = 3 \). Also since \( m = \text{even} \), it is clear that \( G_2 \) is even-sized and \( G_1 \) is odd-sized. By Lemma 3.1, \((G_1,s,t)\) is color-compatible. Since \( l = \text{odd} \geq 1 \) and \( n = \text{even} \geq 4 \), it follows that \( n' - l = \text{odd} \geq 3 \). Moreover, since \( m - k = \text{odd} \geq 3 \), \( n' - l = \text{odd} \geq 3 \), and \( s_t,t_y > m - k \), it is obvious that \((G_1,s,t)\) is not in conditions (F1), (F3), and (F5), and hence \((G_1,s,t)\) is acceptable. In this case, \((G_1,s,t)\) is in Subcase 3.2. The Hamiltonian path in \( (L(m,n,k,l),s,t) \) is obtained similar to Subcase 1.1. Since \( m \geq 6 \), there is at least one edge for combining Hamiltonian cycle and path.

Subcase 2.2.2.2. \( (s_t) = n - 3 \) and \( t_y \leq n - 3 \). Let \( \{G_1,G_2\} \) be \( L \)-shaped separation (type 1) of \( L(m,n,k,l) \) such that \( G_1 = L(m,n,k,l') \), \( G_2 = R(m',n'),l' = n - 4 \), \( m' = k \), \( n' = l - l \), and \( s,t \in G_1 \); (as shown in Fig. 11(d)). Since \( n = \text{odd} \geq 7 \), it follows that \( l = \text{odd} \) and \( n = \text{even} \). Hence, \( G_2 \) is even-sized and \( G_1 \) is odd-sized. By Lemma 3.1, \((G_1,s,t)\) is color-compatible. Since \( n - l' = 4 \), \( m - k = \text{odd} \geq 3 \), and \( s_t,t_y > m - k \), it is clear that \((G_1,s,t)\) is not in conditions (F1), (F3), and (F5), and hence \((G_1,s,t)\) is acceptable. In this case, \((G_1,s,t)\) is in Subcase 2.1. The Hamiltonian path in \( (L(m,n,k,l),s,t) \) is obtained similar to Subcase 1.1; see Fig 11(e). Since \( k = \text{odd} \geq 3 \), thus there is at least one edge for combining Hamiltonian cycle and path.

Subcase 2.2.2.3. \( s_t > n - 3 \) and \( t_y < n - 3 \). This case is similar to Subcase 1.3.1, where \( n' = n - 4 \). Notice that, here, if \( s_t < n - 3 \) and \( t_y > n - 3 \), the role of \( q \) and \( p \) can be swapped (i.e., \( s,p \in G_1 \) and \( q,t \in G_2 \)).

Case 3. \( m = \text{even} \) and \( [(k = 1 \text{ and } n - l = \text{even} > 2) \text{ or } (n - l = \text{odd})] \). Since \( m - k = \text{odd} \geq 3 \), \( n - l > 2 \), and \( k,l \geq 1 \), we have \( m,n \geq 4 \).

Subcase 3.1. \( (l = 1 \text{ and } m - k = 3) \) or \( (l > 1) \).

Subcase 3.1.1. \( s_t,t_y \leq l \). Let \( \{R_1,R_2\} \) be a horizontal separation of \( L(m,n,k,l) \) such that \( R_1 = R(m - k,n') \), \( R_2 = R(m,n - n'),n' = l \), and \( s,t \in R_1 \). Consider Fig. 12(a) and 12(b). Since \( m - k = \text{odd} \) and \( l = \text{odd} \), it follows that \( R_1 \) is odd×odd, and hence \((R_1,s,t)\) is color-compatible. Since \( R_1 \) is odd×odd it is enough to show that \((R_1,s,t)\) is not in condition (F1). To satisfy condition (F1), \( l \) must be 1 and \( s \neq (1,1) \) or \( t \neq (m - k,1) \). The only case that occurs is \( k \times l = 1 \). In this case, \( m - k = 3 \), and it is obvious that \( s = (1,1) \) and \( t = (m - k,1) \). Therefore, \((R_1,s,t)\) is not in condition (F1), and it is acceptable. The Hamiltonian path in \( (L(m,n,k,l),s,t) \) is obtained similar to Subcase 1.1; see Fig. 12(c) and 12(d). Since \( m - k = \text{odd} \geq 3 \), there is at least one edge for combining Hamiltonian cycle and path.

Subcase 3.1.2. \( s_t \leq l \) and \( t_y > l \). This case is similar to Subcase 3.1.1, where \( s,p \in R_1 \), \( q,t \in R_2 \), and \( p \) are
adjacent, and
\[
p = \begin{cases} 
(1, n') & \text{if } s = (1, n') \\
(3, n') & \text{if } s = (1, n') \text{ and } [(n-l > 3) \text{ or } (n-l = 3 \text{ and } t_k \geq 2)]
\end{cases}
\]

Consider Fig. 12(a) and 12(b). A simple check shows that \((R_1, s, p)\) and \((R_2, q, t)\) are color-compatible. \((R_1, s, p)\) does not satisfy conditions (F1) and (F2), the proof is the same as Subcase 3.1.1. Now, consider \((R_2, q, t)\). In this case, \(R_2\) is evenly even or even odd. Since \(m \geq 4\) and \(n - l \geq 3\), it is sufficient to show that \((R_2, q, t)\) is not in condition (F2). The condition (F2) holds, if \(n-l = 3\) and \(q_s > t_k + 1\). The only case that occurs is \(t = (n-1)\), this is impossible because we assume that \(t_k \geq 2\). Therefore, \((R_2, q, t)\) is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. Here, if \(t_k \leq l\) and \(s_r > l\), the role of \(q\) and \(p\) can be swapped (i.e., \(s, p \in R_2\) and \(q, t \in R_1\)).

Now let \(n-l = 3\), \(s_r = 1\), and \(t = (n-1)\). This case is the same as Subcase 1.3.1, where \(s, p \in G_1, q, t \in G_2, p = (m, n-2)\). Consider Fig. 12(e). Since \(n = \text{even}\) implies \(n = \text{even}\) and \(n' = 2\). Thus \(G_2\) is evenly sized and \(G_1\) is odd sized. Obviously \((G_1, s, p)\) and \((G_2, q, t)\) are color-compatible. Consider \((G_1, s, p)\). Since \(n-l = 1\), it is enough to show that \((G_1, s, p)\) is not in conditions (F1) and (F3). Because of \(p = (m, n-2)\) and \(s_r = 1\), it is clear that \((G_1, s, p)\) is not in conditions (F1) and (F3), thus \((G_1, s, p)\) is acceptable. Now, consider \((G_2, q, t)\), In this case, \(G_2\) is evenly even or even odd. Since \(s_r \leq m-k\) and \(q_s = m\), it follows that \((G_2, q, t)\) does not satisfy condition (F1), and hence \((G_2, q, t)\) is acceptable. In this case, \((G_1, s, p)\) and \((G_2, q, t)\) is in Lemma 4.2. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. Here, if \(t = (1, l)\) and \(s = (1, n-1)\), the role of \(q\) and \(p\) can be swapped (i.e., \(s, p \in G_2\) and \(q, t \in G_1\)).

Subcase 3.2. \(n-l = \text{odd}\), \([(k > 1) \text{ or } (k = 1 \text{ and } n-l = 3)]\), and \([s_s, t_s > m-k] \text{ or } (s_s \leq m-k \text{ and } t_s > m-k)\). Assume that \((1, 1)\) is the coordinates of the vertex in the bottom right corner in \((L(m, n, k, l), s, t)\), then This case is similar to Subcase 3.1.

Subcase 3.3. \(m-k = 3\), \(s_s, t_s > l\), and \([(k = 1 \text{ and } n-l = \text{even}) \text{, } (k = 1 \text{ and } n-l = \text{odd}) \text{, } [(n-l = 3 \text{ and } s_s, t_s \leq m-k) \text{ or } (n-l > 3)]\}, \text{ or } (k > 1 \text{ and } s_s, t_s \leq m-k)]\).

Subcase 3.3.1. \(s_s \leq 2\) and \(t_s > 2\). This case is similar to Subcase 3.3.2, where \(G_1 = R(m', n), G_2 = L(m-m', n, k, l), m' = 2, s, p \in G_1, q, t \in G_2, q \text{ and } p \text{ are adjacent, and } p = (2, 1)\). A simple check shows that \((G_1, s, p)\) and \((G_2, q, t)\) are acceptable. In this case, \((G_2, q, t)\) in Lemma 4.2. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2.

Subcase 3.3.2. \(s_s, t_s \leq 2\). Let \(s_s \leq t_s\). This case is the same as Subcase 2.1.2, where \(G_1 = R(2, n'), G_2 = R(2, n-n'), G_3 = L(m-2, n, k, l), n' = \max(s_s, t_s) - 2 \text{ if } \max(s_s, t_s) = n \text{ and } n = \text{even}; \text{ otherwise } n' = \max(s_s, t_s) - 1\). In this case, \(p = (2, 1)\), and \(q = (2, n)\) if \(n = \text{odd}\); otherwise \(q = (2, n-1)\). By the same argument as in proof Subcase 2.1.2, we can easily see that \((G_1, s, p), (G_2, q, t), \text{ and } (G_3, n, z)\) are acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 2.1.1; see Fig. 13(a) and 13(b). Notice that, in this case, if \(t_k < s_r\), the role of \(q\) and \(p\) can be swapped (i.e., \(s, p \in G_2\) and \(q, t \in G_1\)).

Subcase 3.3.3. \(s_s, t_s > 2\). Let \(s_s \leq t_s\). This case is the same as Subcase 2.1.1, where \(G_1 = R(2, n), G_1 = L(m-2, n, k, l), G_2 = R(m-2, n-n), n = \max(s_s, t_s) - 2 \text{ if } \max(s_s, t_s) = n \text{ and } n = \text{odd}; \text{ otherwise } n = \max(s_s, t_s) - 1\).
In this case, $p = (3, 1)$, $q = (3, n - 1)$ if $n = odd$; otherwise $q = (3, n)$. Using the same argument as in proof Subcase 2.1.1, we can easily see that $(G_1, s, p)$, $(G_2, q, t)$, and $(G_3, w, z)$ are acceptable. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1.1; see Fig. 13(c) and 13(d). Notice that, in this case, if $t_i < s_i$, the role of $q$ and $p$ can be swapped (i.e., $s, p \in G_2$ and $q, t \in G_1$).

Subcase 3.4. $m - k > 3$, $n - l = 3$, $s_i, t_i \leq m - k$, and $[l = (1) and [(m - k = 3 \text{ and } s_i,t_i > l) or (m - k > 3)] or (l > 1 and s_i, t_i > l)]$ Let $(1, 1)$ be the coordinates of the vertex in the bottom right corner in $L(m, n, k, l)$, then this case is similar to Subcase 3.3.

Subcase 3.5. $k = 1$, $n - l = even > 2$, $m - k > 3$, and $[(l > 1 \text{ and } s_i, t_i > l) or (l = 1)]$. Since $m - k > 3$ and $k = 1$, we have $m > 6$.

Subcase 3.5.1. $s_i, t_i \leq m - 3$. This case is similar to Subcase 1.3.2, where $G_1 = R(m, n)$, $G_2 = L(m - m', n, k, l)$, $m' = m - 3$, and $s, t \in G_1$. Since $m > 6$, it follows that $m' = odd \geq 3$. Moreover, since $n = odd \geq 5$, we conclude that $G_1$ is odd-sized and $G_2$ is even-sized. By Lemma 3.1, $(G_1, s, t)$ is color-compatible. Because of $m = odd \geq 3$ and $n \geq 5$, it is obvious that $(G_1, s, t)$ is not in conditions (F1) and (F2), and hence it is acceptable. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1; see Fig. 13(e) and 13(f). Notice that the pattern for constructing a Hamiltonian cycle in $G_2$ is shown in Fig. 13(e) and 13(f).

Subcase 3.5.2. $(s_i, t_i > m - 3)$ or $(s_i = m - 3 \text{ and } t_i > m - 3)$. This case is the same as Subcase 3.5.1, where $m = m - 4$, $s, p \in G_1$, and $q, t \in G_2$. Since $m > 6$ implies $m' \geq 2$ and $m = m' = 4$. Hence, $G_1$ is even-sized and $G_2$ is odd-sized. By Lemma 3.1, $(G_2, s, t)$ is color-compatible. Since $m = m - k + 3$ and $n - l > 2$, a simple check shows that $(G_2, s, t)$ is not in conditions (F1), (F3), and (F5), and hence $(G_2, s, t)$ is acceptable. In this case, $(G_2, s, t)$ is in Subcase 3.1.1 or 3.3. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1.

Subcase 3.5.3. $s_i < m - 3 \text{ and } t_i > m - 3$. This case is similar to Subcase 3.5.1, where $m = m - 4$, $s, p \in G_1$, and $q, t \in G_2$ and $p, q$ are adjacent, and $p = (m', 1)$. Since $m = even$, we have $m' = even$. Thus, $G_1$ is even-sized and $G_2$ is odd-sized. Since $m = even$ and $p = (m, 1)$, it is obvious that $p$ is black and $q$ is white. Therefore, $(G_1, s, p)$ and $(G_2, q, t)$ are color-compatible. $(G_2, q, t)$ is not in conditions (F1), (F3), and (F5), the proof is the same as Subcase 3.5.2. Hence, $(G_2, q, t)$ is color-compatible. Now, consider $(G_1, s, p)$. Since $G_1$ is even-sized and $n \geq 5$, it suffices to prove that $(G_1, s, p)$ is not in condition (F1). The condition (F1) holds, if $m' = 2$ and $s_i = p_i \leq n - 1$. Since $p_i = 1$, it is obvious that $(G_1, s, p)$ is not in condition (F1), and hence $(G_1, s, p)$ is acceptable. In this case, $(G_2, q, t)$ is in Subcase 3.1.2. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.2.

Subcase 3.6. $m - k > 3$, $n - l = odd > 3$, and $[(k > 1, s_i, t_i \leq m - k \text{ and } [(l > 1 \text{ and } s_i, t_i > l) or (l = 1)]), (k = 1, l > 1, \text{ and } s_i, t_i > l), or (k \times l = 1)]$. Since $m - k > 3$, $n - l > 3$, and $k \times l \geq 1$, we have $m, n \geq 6$.

Subcase 3.6.1. $s_i, t_i \leq n - 3$ and $[(k > 1) \text{ or } (k = 1 \text{ and } s \neq (m - 1, n - 3) \text{ or } t \neq (m, n - 4))$. This case is the same as Subcase 1.3.1, where $n = n - 3$ and $s, t \in G_1$. Since $m = even$, thus $G_2$ is even-sized and $G_1$ is odd-sized. By Lemma 3.1, $(G_1, s, t)$ is color-compatible. Since $n - l > 3$ and $n = even$, it follows that $n = odd$. Moreover, since $l \geq 1$ implies $n - l = even \geq 2$, and hence $n \geq 3$. Since $n - l = even \geq 2$ and $m - k > 3$, it suffices to prove that $(G_1, s, t)$ is not in conditions (F1) and (F5). The condition (F1) and (F5) hold, if $k = 1, s = (m - 1, n - 3)$, and $t = (m, n - 4)$. By our assumption, This is impossible. Thus, $(G_1, s, t)$ is does not satisfy conditions (F1) and (F5), and hence it is acceptable. In this case, $(G_1, s, t)$ is in Subcase 1.1 or 3.5. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1. Now let $s_i, t_i \leq n - 3, k = 1, s = (m - 1, n - 3)$, and $t = (m, n - 4)$. This case is the same as Subcase 1.3.1, where $n' = n - 4$ and $p = (2, n' + 1)$; see Fig. 14(a) and 14(b). Notice that, here, $(G_1, q, t)$ is in Lemma 4.2, where $t = (m, t + 1)$, or $p = (3, 2)$ or $3.6.2, \text{ where } t_i \neq n - 3$. This case is the same as Subcase 2.2.2.2, where $G_1 = L(m, n, k', l'), G_2 = L(m', n', k', l)$, $k' = m - 3, l' = n - 3, m' = k, n' = l'$, and $s, t \in G_1 \text{ (Fig. 14(c) and 14(d)).}$ Since $m = k = n = even$, it follows that $G_1$ is odd-sized and $G_2$ is even-sized. By Lemma 3.1, $(G_1, s, t)$ is color-compatible. A simple check shows that $(G_1, s, t)$ is not in conditions (F1), (F3) and (F5), thus $(G_1, s, t)$ is acceptable. In this case, $(G_1, s, t)$ is in Subcase 3.1, 3.2, 3.3, or 3.4. Since $m, n \geq 6$, $m' = m - 3$ and $n' = n - 3$, we have $k', l' \geq 3$, Moreover, since $k', l' \geq 3$, there is at least one edge for combining Hamiltonian cycle and path. In this case, the pattern for constructing a Hamiltonian cycle in $G_2$ is shown in Fig. 14(c) and 14(d).

Subcase 3.6.3. $t_i \leq n - 3$ and $s_i > n - 3$.

Subcase 3.6.3.1, $t \neq (1, n - 3)$. This case is similar to Subcase 1.3.1, where $n' = n - 3$. From the proof of Subcase 3.6.1, we know that $G_2$ is even-sized and $G_1$ is odd-sized. We can see easily that $(G_1, q, t)$ and $(G_2, s, p)$ are color-compatible. $(G_1, q, t)$ is not in conditions (F1), (F3), and (F5), the proof is the same as Subcase 3.6.1. Hence, $(G_1, q, t)$ is acceptable. Now, consider $(G_2, s, p)$. Since $n' = n - 3$ and $m \geq 6$, it is sufficient to show that $(G_2, s, p)$
is not in condition (F2). The condition (F2) holds, if \( p_s > s_s + 1 \). Since \( p_s = 1 \), thus \((G_2, q, t)\) does not satisfy condition (F2), and hence it is acceptable. In this case (\( G_1, q, t \)) is in Subcase 1.1, 1.3, or 3.5. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. In this case, if \( s_s \leq n - 3 \) and \( t_r > n - 3 \), the role of \( q \) and \( p \) can be swapped (i.e., \( s, p \in G_1 \) and \( q, t \in G_2 \)).

Subcase 3.6.3.2. \( l = (1, n - 3) \) and \( s_s > n - 3 \) (or \( s = (1, n - 3) \) and \( t_r > n - 3 \)). This case is the same as Subcase 3.6.2. Note that, in this case, \((G_1, s, t)\) is in Subcase 3.1.2.

**Lemma 4.4.** Suppose that \( P(L(m, n, k, l), s, t) \) is an acceptable Hamiltonian path problem with \( m - k = even \). Let \( L(m, n, k, l) \) be even-sized. Then there is an acceptable separation for \((L(m, n, k, l), s, t)\) and it has a Hamiltonian path.

**Proof.** We have one of the following five cases and in each of them we will show that \((L(m, n, k, l), s, t)\) has an acceptable separation and has a Hamiltonian path. Note that, here, \( s \) and \( t \) have different colors. Let \( m \times n = even \times even \) or \( even \times odd \) (resp. \( m \times n = odd \times odd \)). Since \((L(m, n, k, l)\) is even-sized and \( m - k = even \), it follows that \( k \times t = even \times even \) or \( even \times odd \) (resp. \( k \times l = odd \times odd \)).

Case 1. \( m - k = 2 \) and \( n - l \geq 2 \). Since \( m - k = 2, n - l \geq 2, \) and \( k, l \geq 1, \) we have \( m, n \geq 3 \).

Subcase 1.1. \( s_s, t_t \leq l \). This case is the same as Subcase 3.1.1 of Lemma 4.3. Since \( m - k = 2 \) it follows that \( R_1 \) is even-sized. Moreover, since \((L(m, n, k, l)\) is even-sized, we conclude that \( R_2 \) also is even-sized. Thus, \((R_1, s, t)\) is color-compatible. Since \( m - k = 2 \), it suffices to prove \((R_1, s, t)\) is not in condition (F1). The condition (F1) holds, if \( s_s = t_t \neq 1 \). If \( s_s = t_t \neq 1 \), then \((L(m, n, k, l), s, t)\) satisfies condition (F1), which contradicts the fact that \((L(m, n, k, l), s, t)\) is acceptable. Therefore, \((R_1, s, t)\) is not in condition (F1), and hence it is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3.

Subcase 1.2. \( s_s, t_t > l \) and \( (n - l > 2) \) and \( (s \text{ or } t) \neq (1, l + 1)) \) or \((n - l = 2)\). This case is similar to Subcase 3.1.1 of Lemma 4.3, where \( s, t \in R_2 \). From the proof of Subcase 1.1, we know that \( R_1 \) is even-sized. Hence, \((R_2, s, t)\) is color-compatible. \((R_2, s, t)\) is not in conditions (F1) and (F2), because it satisfies condition (F1) or (F2), then \((L(m, n, k, l), s, t)\) satisfies condition (F1) or (F8), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Therefore, it follows that \((R_2, s, t)\) is not in conditions (F1) and (F2), and hence it is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3. Consider Fig. 15(a). Since \( s \neq w \) and \( t \neq w \), thus in the Hamiltonian path of \( R_2 \) the edge \( e \) must exist, hence we can combine the path of \( R_2 \) with the cycle of \( R_1 \) (as shown in Fig. 15(b)).

Subcase 1.3. \( n - l > 2 \), \( s_s, t_t > l \), and \( s \text{ or } t \) \( = (1, l + 1) \). Let \( s = (1, l + 1) \). Let \((G_1, G_2)\) be a \( L \)-shaped separation (type III) of \((L(m, n, k, l)\) such that \( V(G_1) = \{x = 1, y = l + 1 \leq x \leq 2.1 \leq y \leq l\}, G_2 = L(m, n, k, l) \setminus G_1, s, p \in G_1, q, t \in G_2, p \) and \( q \) are adjacent, and \( p = (2, l) \). Consider Fig. 15(c)–(g). Clearly, \( G_1 \) and \( G_2 \) are odd-sized. In this case, \( G_2 = L(m, n', 1, 1) \), where \( n' = n - l \). We can easily see that \((G_1, s, p)\) and \((G_2, q, t)\) are color-compatible. Consider \((G_1, s, p)\), we can easily see that \((G_1, s, p)\) is not in conditions (F1), (F3), and (F5). Now, consider \((G_2, q, t)\). Since \( n - l > 2 \) and \( l = 1 \), it follows that \( n' - l \geq 2 \). Moreover, since \( m \geq 3 \) and \( k = 1 \), we have \( m - k \geq 2 \). Since \( m - k \geq 2 \) and \( n' - l \geq 2 \), it is enough to show that \((G_2, q, t)\) is not in conditions (F1) and (F5). Since \( q = (2, l + 1) \), the only case that occurs is \( m = 3 \) and \( t = (m, l + 2) \). In this case, \((L(m, n, k, l), s, t)\) satisfies condition (F7), which is contradiction because \((L(m, n, k, l), s, t)\) is acceptable. So, \((G_2, q, t)\) is not in conditions (F1) and (F5), and hence it is acceptable. In this case, \((G_1, s, p)\) is in Subcase 4.2 and \((G_2, q, t)\) is in Case 3 of Lemma 4.3. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2.
Subcase 1.4. \( s_j \leq l \) and \( t_j > l \). This case is the same as Subcase 3.1.2 of Lemma 4.3, where
\[
p = \begin{cases} 
(1,n'); & \text{if } s \text{ and } w \text{ have different colors} \\
(2,n'); & \text{otherwise}
\end{cases}
\]
and \( w = (1,n') \). Clearly, \((R_1,s,p)\) and \((R_2,q,t)\) are color-compatible. Consider \((R_1,s,p)\). Since \( m-k = 2 \), \( p_j = n' \), and \( s_j \leq n' \), it is clear that \((R_1,s,t)\) is not in conditions (F1) and (F2). Thus, \((R_1,s,p)\) is acceptable. Now, consider \((R_2,q,t)\). The condition (F1) holds, if \( n-l = 2 \) and \( t = (2,k) \). In this case, \((L(m,n,k,l),s,t)\) satisfies condition (F6), this contradicts the assumption that \((L(m,n,k,l),s,t)\) is acceptable. Thus, \((R_2,q,t)\) is not in condition (F1). The condition (F2) occurs, when
(i) \( n-l = 3 \), \( t_j > m-k+1 \), and \( t \) is black (when \( n = \text{even} \)) or \( t \) is white (when \( n = \text{odd} \)). Since \((L(m,n,k,l),s,t)\) is acceptable, the only case that occurs is \( t = (m-k+1,l+1) \) or \( t = (m-k+1,n) \). In this case, \( q = (m-k,l+1) \) and clearly \((R_2,q,t)\) is not in condition (F2); or
(ii) \( m = 3 \), \( n-l > 2 \), \( t_j > l+1 \), and \( t \) is black. Since \((L(m,n,k,l),s,t)\) is acceptable, the only case that occurs is \( t = (1,l+1) \) or \( t = (m-l+1) \). In this case, \( q = (2,l+1) \) and clearly \((R_2,q,t)\) is not in condition (F2).
Therefore, \((R_2,q,t)\) is acceptable. The Hamiltonian path in \((L(m,n,k,l),s,t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. Notice that, here, if \( t_j \leq l \) and \( s_j > l \), the role of \( q \) and \( p \) can be swapped (i.e., \( s, p \in R_2 \) and \( q, t \in R_1 \)).

Case 2. \( m-k > 2 \) and \( n-l = 2 \). Let \((1,1)\) be the coordinates of the vertex in the bottom right corner in \((L(m,n,k,l),s,t)\), then using the same argument as in the proof Case 1 the Hamiltonian \((s,t)\)-path is obtained.

Case 3. \( k > 1 \), \( m-k > 2 \), and \( n-l > 2 \). In this case, since \( m-k > 2 \), \( n-l > 2 \), \( k \geq 2 \), and \( l \geq 1 \), it follows that \( m \geq 6 \) and \( n \geq 4 \).

Subcase 3.1. \( s_j \leq m-k \). This case is similar to Subcase 1.1 of Lemma 4.3. Consider Fig. 16. In this case \( R_1 \) is even-\(n \text{ even} \) or even-\(n \text{ odd} \). Obviously, \((R_1,s,t)\) is color-compatible. Since \( m-k > 2 \) and \( n \geq 4 \), it is clear that \((R_1,s,t)\) is not in conditions (F1) and (F2). Therefore, \((R_1,s,t)\) is acceptable. The Hamiltonian path in \((L(m,n,k,l),s,t)\) is obtained similar to Subcase 1.1 of Lemma 4.3.

In the following, we describe combining a Hamiltonian path in \( R_1 \) with the constructed cycle in \( R_2 \). Any Hamiltonian path \( P \) in \( R_1 \) contains all the vertices of \( R_1 \). Therefore, \( P \) should contain a boundary edge of \( R_1 \) that has a parallel
edge in $R_2$, except when $n - l = 3$, in this case $R_1$ may have no boundary edge parallel to any edge of $R_2$ (see Fig. 17(a)). But in this case, $s (or t) = (m - k, n - 1)$ and $t (or s) = (m - k, n)$. Let an edge $(t, v)$ (or $(s, v)$) of $R_1$ be adjacent to $R_2$. Then $P$ must contain an edge $(t, u)$ (or $(s, u)$) such that $u$ is not adjacent to $R_2$ as depicted in Fig. 17(b). Hence, replacing the role of $u$ with $v$ we can modify $P$ such that it contains a boundary edge adjacent to $R_2$ (see Fig. 17(c)). Using two parallel edges of $P$ and the Hamiltonian cycle of $R_2$ such as the two darkened edges of Fig. 17(c) we can combine them; see Fig. 17(d).

Subcase 3.2. $s, t > m - k$ and $k > 3$. This case is similar to Subcase 1.1 of Lemma 4.3, where $s, t \in R_2$. Consider Fig. 16. In this case, $R_2$ is even×even, even×odd, or odd×even. Since $k > 3$ and $n - l \geq 3$, it suffices to prove that $(R_2, s, t)$ is not in condition (F2). It is clear that if $(R_2, s, t)$ satisfies condition (F2), then $(L(m, n, k, l), s, t)$ satisfies condition (F9), which is contradiction because $(L(m, n, k, l), s, t)$ is acceptable. Thus, it follows that $(R_2, s, t)$ does not satisfy condition (F2), and hence it is acceptable. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.3.

Subcase 3.3. $s, t > m - k$ and $k \leq 3$. This case is the same as Subcase 1.3.2 of Lemma 4.3, where $G_1 = R(m', n)$, $G_2 = L(m - m', n, k, l)$, $m' = m - k - 2$ and $s, t \in G_2$. Let $m = \text{even}$ (resp. $m = \text{odd}$) and $k = 2$ (resp. $k = 3$), then $m - k - 2 = \text{even}, m - m' = 4$ and $m - m' - k = 2$ (resp. $m - k - 2 = \text{even}, m - m' = 5$, and $m - m' - k = 2$). Hence, $G_1$ and $G_2$ are even-sized. By Lemma 3.1, $(G_2, s, t)$ is color-compatible. Since $m - m' - k = 2$ and $n - l \geq 3$, it suffices to prove that $(G_2, s, t)$ is not in condition (F8). The condition (F8) occurs, when $n - l = 3$ and $s = t = n - 1$. In this case, $(L(m, n, k, l), s, t)$ satisfies condition (F9), this contradicts the assumption that $(L(m, n, k, l), s, t)$ is acceptable. Therefore, $(G_2, s, t)$ is not in condition (F8), and hence it is acceptable. In this case, $(G_2, s, t)$ is in Subcase 1.1. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.3.

Subcase 3.4. $s, t \leq m - k$ and $l > m - k$. This case is similar to Subcase 1.3 of Lemma 4.3, where

$$p = \begin{cases} (m - k, l + 1); & \text{if } n - l = \text{even and } t \text{ and } w \text{ have different colors} \\ (m - k, l + 2); & \text{if } n - l = \text{odd, } t \text{ and } w \text{ have the same color, and } [(k \neq 2) \text{ or } (k = 2, n - l > 3, \text{ and } t \neq (m, l + 2))] \\ (m - k, l + 4); & \text{if } n - l = \text{odd > } 3, t \text{ and } w \text{ have the same color, } k = 2, \text{ and } t = (m, l + 2) \\ (m - k, n); & \text{otherwise} \end{cases}$$

and $w = (m - k + 1, l + 1)$. Consider Fig. 16. Clearly, $(R_1, s, p)$ and $(R_2, q, t)$ are color-compatible. $(R_1, s, p)$ is not in conditions (F1) and (F2), the proof is the same as Subcase 3.1. Now, consider $(R_2, q, t)$. To satisfy condition (F1), $k$ must be $2$, $n - l > 3$, and $l + 2 \leq q, t = n - 1$. A simple check shows that $(R_2, q, t)$ is not in condition (F1). $(R_2, q, t)$ is not in condition (F2), the proof is the same as Subcase 1.4. Thus, $(R_1, s, p)$ and $(R_2, q, t)$ are acceptable. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.2.

Case 4. $k = 1, l > 1, m - k > 2$, and $n - l > 2$. Let $(1, 1)$ be the coordinates of the vertex in the bottom right corner in $(L(m, n, k, l))$, then using the same argument as in Case 3 the Hamiltonian $(s, t)$–path is obtained.

Case 5. $k = 1, l = 1, m - k > 2$, and $n - l > 2$.

Subcase 5.1. $m = n = 5$ and $[(s, t) \leq m - k]$ or $[(s = 1 \text{ and } t = m)].$

Subcase 5.1.1. (i) $s, t \leq 3$ and $[s$ is black and $[(s = 2 \text{ and } t > s) (s = 2 \text{ odd and } t > s + 1)]$ or $(s$ is white, $s = (m - 2, 1), \text{ and } t = (m - 1, 3)])$; (ii) $s, t > 3$, and $2 \leq s = t \leq m - 1$; or (iii) $s > 3, t \leq 3$, $s = (2, 5)$, and $t = (3, 1)$. The Hamiltonian paths of all the possible problems in this case are depicted in Fig. 18. Fig. 19(a) and 19(b).

Subcase 5.1.2. $s, t > 3$ and $[(s = t = 1 \text{ or } m) \text{ or } (s \neq t)]$. This case is the same as Subcase 1.3.1 of Lemma 4.3, where $n' = 3$ and $s, t \in G_2$. Since $n = 5$ and $n' = 3$, we have $n - n' = 2$. Thus, $G_2$ is odd×even and $G_1$ is
even-sized. By Lemma 3.1, \((G_2, s, t)\) is color-compatible. A simple check shows that \((G_2, s, t)\) is not in conditions (F1) and (F2), hence \((G_2, s, t)\) is acceptable. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3; see Fig. 19(c). Since \(m = 5\), thus there is at least one edge for combining Hamiltonian cycle and path. In this case, the pattern for constructing a Hamiltonian cycle in \(G_1\) is shown in Fig. 19(c).

Subcase 5.1.3. \(s_x, t_y \leq 3\) and \([(s \text{ is black}, s_y = \text{odd}, \text{and } t_x = s_x + 1) \text{ or } (s \text{ is white and } t_x \geq s_x)\]. This case is similar to Subcase 5.1.2, where \(s, t \in G_1\). A simple check shows that \((G_1, s, t)\) is acceptable. In this case, since \(n' - l = 2\), thus \((G_1, s, t)\) is in Case 2. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 5.1.2.

Subcase 5.1.4. \(t_y \leq 3\) and \(s_x > 3\). This case is similar to Subcase 5.1.2, where \(q, t \in G_1, s, p \in G_2, p\) and \(q\) are adjacent,

\[
\begin{align*}
(1, 4); & \text{ if } t \text{ and } w \text{ have different colors} \\
(2, 4); & \text{ if } t \text{ and } w \text{ have the same color and } s \neq (2, 5) \\
(4, 4); & \text{ if } t \text{ and } w \text{ have the same color, } s = (2, 5), \text{ and } t \neq (3, 1)
\end{align*}
\]

and \(w = (1, 3)\). It is clear that \((G_1, q, t)\) and \((G_2, s, p)\) are color-compatible. Consider \((G_2, s, p)\). From the proof of Subcase 5.1.2, we know that \(G_2\) is odd×even. Since \(m = 5\), \((G_2, s, p)\) is not in the conditions (F2). The condition (F1) holds, if \(n - n' = 2\) and \(2 \leq s_x = p_x \leq 4\). A simple check shows that \(s_x = p_x = 1\) or \(s_x \neq p_x\), and hence \((G_2, s, p)\) is acceptable. Now consider \((G_1, q, t)\). Since \(n' = 3\), \(m - 1 = \text{even} \geq 2\), and \(n - l = 2\), it suffices to prove \((G_1, q, t)\) does not satisfy conditions (F1), (F7), and (F8). The condition (F1) holds if \(t = (4, 2)\) and \(q = (4, 3)\). In this case, \((L(m, n, k, l), s, t)\) satisfies condition (F4), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Thus, \((G_1, q, t)\) is not in condition (F1). The condition (F8) occurs, when \(q = (2, 3)\) and \(t_x > q_x + 1\). Since \(s_x, t_x \leq m - k\), the only case that occurs is \(t = (4, 2)\). In this case \((L(m, n, k, l), s, t)\) satisfies condition (F4), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Hence, \((G_1, q, t)\) does not satisfy condition (F8). The condition (F7) holds, if \(t = (3, 1)\) and \(q = (4, 3)\). Clearly, this case does not occur. So, \((G_1, q, t)\) is not in conditions (F1), (F7), and (F8), and hence it is acceptable. In this case, \((G_1, q, t)\) is in Case 2. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2 (Fig. 19(d)).

Subcase 5.2. \(m = n = 5\) and \([s_x \leq m - k, s_y > l, \text{ and } t_x = m]\) or \(([s_x = t_x = m]\). Assume that \((1, 1)\) is the coordinates of the vertex in the bottom right corner in \((L(m, n, k, l)\), then using the same argument as in Subcase 5.1 the Hamiltonian \((s, t)\)–path is obtained.

Case 5.3. \(m > 5 \text{ or } n > 5\).

Subcase 5.3.1. \(m > 5 \text{ and } n = 5\).

Subcase 5.3.1.1. \(s_x, t_y \leq m - 3\). This case is similar to Subcase 1.3.2 of Lemma 4.3, where \(G_1 = R(m', n)\), \(G_2 = L(m - m', n, k, l)\), \(m' = m - 3\) and \(s, t \in G_1\). Since \(m > 5\), thus \(m' = \text{even} \geq 4\), \(G_1\) is even×odd, and \(G_2\) is

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**Figure 18.** A Hamiltonian \((s, t)\)–path in \(L(5, 5, 1, 1)\).

**Figure 19.** A Hamiltonian \((s, t)\)–path in \(L(m, n, k, l)\).
even-sized. By Lemma 3.1, \((G_1, s, t)\) is color-compatible. Since \(m' \geq 4\) and \(n = 5\), it is clear that \((G_1, s, t)\) is not in conditions (F1) and (F2), hence \((G_1, s, t)\) is acceptable. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3 (Fig. 19(e)). In this case, the pattern for constructing a Hamiltonian cycle in \(G_1\) is shown in Fig. 19(e).

Subcase 5.3.1.2. \(s_t, l_t > m - 3\). This case is similar to Subcase 5.3.1.1, where \(m' = m - 5\) and \(s, t \in G_2\). Since \(m > 5\), it follows that \(m' = \text{even} \geq 2\) and \(m - m' - k = 5\). Therefore, \(G_1\) and \(G_2\) are even-sized. By Lemma 3.1, \((G_2, s, t)\) is color-compatible. Since \(n = 5\), thus \(n - l = 4\). Moreover, since \(m - m' - k = 4\) and \(n - l = 4\), it suffices to prove that \((G_2, s, t)\) is not in condition (F4). It is clear that if \((G_2, s, t)\) satisfies condition (F4), then \((L(m, n, k, l), s, t)\) satisfies condition (F4), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. So, \((G_2, s, t)\) is not in condition (F4), and hence it is acceptable. In this case, \((G_1, s, t)\) is in Subcase 5.1. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3.

Subcase 5.3.1.3. \(s_t \leq m - 3\) and \(l_t > m - 3\). This case is the same as Subcase 5.3.1.1, where \(s, p \in G_1\), \(q, t \in G_2\), \(p\) and \(q\) are adjacent, and \(p = (m - 3, 2)\) if \(s\) is black; otherwise \(p = (m - 3, n)\). Clearly \((G_1, s, p)\) and \((G_2, q, t)\) are color-compatible. \((G_1, s, p)\) is not in conditions (F1) and (F2), the proof is the same as Subcase 5.3.1.1. Thus, \((G_1, s, p)\) is acceptable. Now, consider \((G_2, q, t)\). Since \(m - m' = 3\), \(m - m' - k = 2\), and \(n - l = 4\), it suffices to prove that \((G_2, q, t)\) is not in conditions (F1) and (F8). The condition (F1) holds, \(t = (m - 1, 2)\). In this case, \((L(m, n, k, l), s, t)\) satisfies condition (F4), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Therefore, \((G_2, q, t)\) is not in condition (F1). The condition (F8) occurs, when \((i)\) \(t\) is white and \(q_t > t + 1\), or \((ii)\) \(t\) is black and \(q_t < t - 1\). Since \(q_t = 2\), where \(t\) is white, and \(q_t = n\), where \(t\) is black, it is obvious that \((G_2, q, t)\) is not in condition (F8), and hence it is acceptable. In this case, \((G_2, q, t)\) is in Case 1. The Hamiltonian path in \((L(m, n), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2.

Subcase 5.3.2. \(m = 5\) and \(n > 5\). Suppose that \((1, 1)\) is the coordinates of the vertex in the bottom right corner in \((L(m, n, k, l), s, t)\), then using the same argument as in Subcase 5.3.1. the Hamiltonian path \((s, t)\)–path is obtained.

Subcase 5.3.3. \(m > 5\) and \(n > 5\). This case is the same as Subcase 5.3.1 (Fig. 19(f)). Notice that, in this case, if \(s_t, l_t > m - 3\), then \((G_2, s, t)\) is in Subcase 5.3.2. \(m = 5\) and \(n > 5\). Suppose that \((1, 1)\) is the coordinates of the vertex in the bottom right corner in \((L(m, n, k, l), s, t)\), then using the same argument as in Subcase 5.3.1. the Hamiltonian path \((s, t)\)–path is obtained.

Lemma 4.5. Suppose that \(P(L(m, n, k, l), s, t)\) is an acceptable Hamiltonian path problem with \(m = k = 5\) and \(n - l = \text{odd} \geq 3\). Let \((L(m, n, k, l), s, t)\) be even-sized. Then there is an acceptable separation for \((L(m, n, k, l), s, t)\) and it has a Hamiltonian path.

Proof. Note that, here, \(m \times n = \text{even} \times \text{odd}\) and \(k \times l = \text{odd} \times \text{even}\). Since \(m = k = \text{odd} \geq 3\), \(k \geq 1\), \(n - l = \text{odd} \geq 3\), and \(l = \text{even} \geq 2\), it follows that \(m \geq 4\) and \(n \geq 5\). Consider the following cases for \(s\) and \(t\). In all cases, first we prove that \((L(m, n, k, l), s, t)\) has an acceptable separation, next we show that \((L(m, n, k, l), s, t)\) has Hamiltonian path.

Case 1. \(s_t, t_s \leq l\).

Subcase 1.1. \((l > 2)\) or \((l = 2)\). [as \(s_t, t_s \leq l\)]. This case is similar to Subcase 3.1.1 of Lemma 4.3. Consider Fig. 20(a). Since \(R_1\) is odd\(\times\)even and \(s, t \in R_1\), it is obvious that \((R_1, s, t)\) is color-compatible. The condition (F1) holds, if \(l = 2\) and \(2 \leq t_s, t_s = s_t \leq m - k - 1\). Since \(s_t \neq t_s\) or \(s_t = t_s = 1 \text{ or } m - k\), clearly \((R_1, s, t)\) is not in condition (F1). It is obvious that if \((R_1, s, t)\) satisfies condition (F2), then \((L(m, n, k, l), s, t)\) satisfies condition (F9), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Therefore, it follows that \((R_1, s, t)\) is not in condition (F2), and hence \((R_1, s, t)\) is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3. Since \(m - k = \text{odd} \geq 3\), there is at least one edge for combining Hamiltonian cycle and path.

Subcase 1.2. \(m - k > 3\) and \(2 \leq s_t, l_s \leq m - k - 2\). Let \((G_1, G_2)\) be a \(L\)-shaped separation (type 1) of \((L(m, n, k, l), s, t)\) such that \(G_1 = L(m, n, k', l), k' = k + 1, G_2 = R(m, n', l), m' = m - k' - k, n' = l\), and \(s, t \in G_1\). Consider Fig. 20(b). Since \(l = \text{even}\), \(G_2\) is even-sized and hence \(G_1\) is even-sized. By Lemma 3.1, \((G_1, s, t)\) is color-compatible. Since \(m - k = \text{odd} > 3\) implies \(m \geq 6\) and \(m - k' = \text{even} \geq 4\). Moreover since \(n - l \geq 3\) and \(m - k' \geq 4\), it suffices to prove that \((G_1, s, t)\) is not in condition (F9). The condition (F9) holds, if \(n - l = 3\) and \(s_t > m - k\) or \(t_s > m - k\). Since \(s_t, t_s \leq l\), it is clear that \((G_1, s, t)\) is not in condition (F9), and hence it is acceptable. In this case \((G_1, s, t)\) is in Subcase 3.1 of Lemma 4.4. Notice that, in this case, \(G_2\) is a one-rectangle. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.2 of Lemma 4.4; see Fig. 20(b).

Subcase 1.3. \(s_t = t_s = m - k - 1\). This case is similar to Subcase 1.2, where \(k' = k + 2\) and \(s, t \in G_2\). Consider Fig. 20(c). It is obvious that \((G_2, s, t)\) is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.3. In this case, the pattern for constructing a Hamiltonian cycle in \(G_2\) is shown in Fig. 20(c).
Figure 20. (a) a horizontal separation of $L(m, n, k, l)$ and (b)-(e) a Hamiltonian $(s, t)$-- path in $L(m, n, k, l)$.

Figure 21. (a) a horizontal separation of $L(m, n, k, l)$ and (b)-(e) a Hamiltonian $(s, t)$-- path in $L(m, n, k, l)$.

Case 2. $s_j, t_j > l$.

Subcase 2.1. $(n - l \geq 5)$ or $(m - k = n - l = 3)$. This case is similar to Subcase 1.1, where $s, t \in R_2$. Consider Fig. 20(a). Here, $R_2$ is evenodd. Obviously, $(R_2, s, t)$ is color-compatible. Since $m \geq 4$ and $n - l \geq 3$, it suffices to prove that $(R_2, s, t)$ is not in condition (F2). If $(R_2, s, t)$ satisfies condition (F2), then $(L(m, n, k, l)$ satisfies condition (F9), this contradicts the assumption that $(L(m, n, k, l), s, t)$ is acceptable. So, $(R_2, s, t)$ is not in condition (F2), and hence it is acceptable. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.3.

In the following, we describe combining a Hamiltonian path in $R_2$ with the constructed cycle in $R_1$. Any Hamiltonian path $P$ in $R_2$ contains all the vertices of $R_2$. Therefore, $P$ should contain a boundary edge of $R_2$ that has a parallel edge in $R_1$, except when $m - k = 3$, in this case $R_2$ may have no boundary edge parallel to any edge of $R_1$ (see Fig. 20(d)). But in this case, $s = (1, l + 1)$ and $t = (2, l + 1)$. In this case for constructing a Hamiltonian path in $R_2$ by the algorithm in [2] (Fig. 20(e)), first we combine a Hamiltonian path in $R_{25}$ with a Hamiltonian cycle in $R_{24}$ (this subpath is called $P_{1}$) (Fig. 21(a)), then we combine $P_{1}$ with a Hamiltonian cycle in $R_{22}$ using two parallel edges $e_1$ and $e_2$ as shown in Fig. 21(a). Finally, using two parallel edges of $P$ and the Hamiltonian cycle of $R_1$ such as the two darkened edges of Fig. 21(b) we can combine them; see Fig. 21(c).

Subcase 2.2. $m - k \geq 5$ and $n - l = 3$.

Subcase 2.2.1. $l = 2$. This case is similar to Subcase 1.2. Notice that, in this case, it is clear that $(G_1, s, t)$ is not in condition (F9). Because if $(G_1, s, t)$ satisfies condition (F9), then $(L(m, n, k, l), s, t)$ satisfies condition (F9), this contradicts the assumption that $(L(m, n, k, l), s, t)$ is acceptable. Therefore, it follows that $(G_1, s, t)$ is not in condition (F9), and hence it is acceptable.

Subcase 2.2.2. $l > 2$. This case is similar to Subcase 1.3.1 of Lemma 4.3, where $G_1 = R(m - k, n')$, $G_2 = L(m, n - n', k, l)$, $n' = l - 2$, $l = 2$, and $s, t \in G_2$; see Fig 21(d). Since $l = even$ implies $n = even \geq 2$. Thus, $G_1$ and $G_2$ are even-sized. By Lemma 3.1, $(G_2, s, t)$ is color-compatible. $(G_2, s, t)$ is not in condition (F9), the proof is the same as Subcase 2.2.1. Thus, $(G_2, s, t)$ is acceptable. In this case, $(G_2, s, t)$ is in Subcase 2.2.1. The Hamiltonian path in $(L(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.3. Since $m - k = odd \geq 5$, thus there is at least one edge for combining Hamiltonian cycle and path.
Case 3. \( s_y \leq l \) and \( t_y > l \). This case is similar to Subcase 3.1.2 of Lemma 4.3, where

\[
p = \begin{cases} 
(1, n'); & \text{if } s \text{ and } w \text{ have different colors} \\
(m - k - 1, n'); & \text{if } s \text{ and } w \text{ have the same color and } [(l \neq 2) \text{ or } (l = 2, m - k > 3, \text{ and } s \neq (m - k - 1, 1))] 
\end{cases}
\]

and \( w = (1, n') \). As shown in Fig. 20(a), \((R_1, s, p)\) is odd\(x\)even and \((R_2, q, t)\) is even\(x\)odd. It is clear that \((R_1, s, p)\) and \((R_2, q, t)\) are color-compatible. Consider \((R_1, s, p)\). The condition (F1) holds, if \( l = 2 \) and \( s_y = p_x \neq 1 \) or \( m - k \). The only case that occurs is \( s_y = p_x = m - k - 1 \). Since \( s_y \neq (m - k - 1, 1) \), thus \((R_1, s, p)\) is not in condition (F1). Clearly to satisfy condition (F2), \( m - k \) must be 3 and \( s_y < l \) and \( s \) is black. Since \((L(m, n, k, l), s, t)\) is acceptable, the only case that occurs is \( s = (1, 1) \) or \( s = (m - k, l) \). In this case, \( p = (m - k - 1, l) \) and \((R_1, s, p)\) is not in condition (F2). Hence, \((R_1, s, p)\) is acceptable. Now, consider \((R_2, q, t)\). To satisfy condition (F1), \( m \) must be 2. Since \( m \geq 4 \), this case cannot occur. The condition (F2) occurs, when \( n - l = 3 \), \( t_y > m - k \), and \( t \) is white. In this case, \((L(m, n, k, l), s, t)\) satisfies condition (F9), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. So, \((R_2, q, t)\) does not satisfy condition (F2), and hence it is acceptable. The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. Notice that, here, if \( t_y \leq l \) and \( s_y > l \), the role of \( q \) and \( p \) can be swapped (i.e., \( s, p \in R_2 \) and \( q, t \in R_1 \)).

Now suppose that \( m - k > 3 \), \( l = 2 \), \( s = (m - k - 1, 1) \), and \( t_y > l \). This case is the same as Subcase 1.3, where \( s, p \in G_2 \), \( q, t \in G_1 \), and \( p \) and \( q \) are adjacent, \( p = (m - k - 1, 2) \), and \( q = (m - k - 1, 3) \). Consider Fig. 21(e). In this case, \( G_1 \) is even-sized and \( G_2 \) is \( 2 \times 2 \). Clearly \((G_1, s, p)\) and \((G_2, q, t)\) are color-compatible. Consider \((G_1, q, t)\). Since \( m - k' = odd \geq 3 \) and \( n - l = odd \geq 3 \), it suffices to prove \((G_1, q, t)\) is not in condition (F9). The condition (F9) holds, if \( n - l = 3 \), \( t \) is white, and \( t_y > m - k \). In this case, \((L(m, n, k, l), s, t)\) satisfies condition (F9), this contradicts the assumption that \((L(m, n, k, l), s, t)\) is acceptable. Therefore, it follows that \((G_1, q, t)\) is not in condition (F9), and hence \((G_1, q, t)\) is acceptable. In this case, \((G_1, q, t)\) is in Case 2. Clearly, \((G_2, s, p)\) is not in conditions (F1) and (F2). The Hamiltonian path in \((L(m, n, k, l), s, t)\) is obtained similar to Subcase 1.1 of Lemma 4.2. Notice that, here, if \( t = (m - k - 1, 1) \), and \( s_y > l \), the role of \( q \) and \( p \) can be swapped (i.e., \( s, p \in G_1 \) and \( q, t \in G_2 \)).

Now, we prove Theorem 4.1.

**Proof.** Consider the following cases.

Case 1. \( m - k = 1 \) or \( n - l = 1 \). \((L(m, n, k, l), s, t)\) is in Lemma 4.2.

Case 2. \( m - k, n - l > 1 \) and \((L(m, n, k, l), s, t)\) is odd-sized. Notice that, here, \( m - k = odd \).

Subcase 2.1. \( n = odd \) and \( k > 1 \).

Subcase 2.1.1. \( s_y, t_x \leq m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 1.1 of Lemma 4.3.

Subcase 2.1.2. \( s_y, t_x > m - k \).

Subcase 2.1.2.1. \( n - l = odd \). \((L(m, n, k, l), s, t)\) is in Subcase 1.2 of Lemma 4.3.

Subcase 2.1.2.2. \( n - l = even \). \((L(m, n, k, l), s, t)\) is in Subcase 2 of Lemma 4.2.

Subcase 2.1.3. \( s_y \leq m - k \) and \( t_x > m - k \).

Subcase 2.1.3.1. \( n - l = odd \). \((L(m, n, k, l), s, t)\) is in Subcase 1.2 or 1.3 of Lemma 4.3.

Subcase 2.1.3.2. \( n - l = even \). \((L(m, n, k, l), s, t)\) is in Subcase 1.3 of Lemma 4.3.

Subcase 2.2. \( n = odd \) and \( k = 1 \).

Subcase 2.2.1. \( n - l = 2 \).

Subcase 2.2.1.1. \( s_y, t_x \leq m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 1.1 of Lemma 4.3.

Subcase 2.2.1.2. \( s_y, t_x > m - k \). By Theorem 3.2, this case cannot occur.

Subcase 2.2.1.3. \( s_y \leq m - k \) and \( t_x > m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 1.3 of Lemma 4.3.

Subcase 2.2.2. \( n - l > 2 \).

Subcase 2.2.2.1. \( s_y, t_x \leq l \). \((L(m, n, k, l), s, t)\) is in Subcase 3.1.1 or 3.5 of Lemma 4.3.

Subcase 2.2.2.2. \( s_y, t_x > l \). \((L(m, n, k, l), s, t)\) is in Subcase 3.3 or 3.5 of Lemma 4.3.

Subcase 2.2.2.3. \( s_y \leq l \) and \( t_x > l \). \((L(m, n, k, l), s, t)\) is in Subcase 3.1.2 or 3.5 of Lemma 4.3.

Subcase 2.3. \( n = even \). In this case, \( n - l = odd \).

Subcase 2.3.1. \( s_y, t_x \leq l \). \((L(m, n, k, l), s, t)\) is in Subcase 3.1.1, 3.4, or 3.6.1 of Lemma 4.3.

Subcase 2.3.2. \( s_y, t_x > l \). \((L(m, n, k, l), s, t)\) is in Subcase 3.2, 3.3, 3.4, or 3.6 of Lemma 4.3.
Subcase 2.3.3 \( s_y \leq l \) and \( t_y > l \) (\( s_y \leq l \) and \( s_y > l \)). \((L(m, n, k, l), s, t)\) is in Subcase 3.1.2, 3.4, 3.6.1, or 3.6.3 of Lemma 4.3.

Case 3. \( m - k, n - l > 1 \) and \((L(m, n, k, l))\) is even-sized.

Subcase 3.1. \( m - k = \text{even} \).

Subcase 3.1.1. \( m = k = 2 \) and \( n - l \geq 2 \).

Subcase 3.1.1.1. \( s_y, t_y \leq l \). \((L(m, n, k, l), s, t)\) is in Subcase 1.1 of Lemma 4.4.

Subcase 3.1.1.2. \( s_y, t_y > l \). \((L(m, n, k, l), s, t)\) is in Subcase 1.2 or 1.3 of Lemma 4.4.

Subcase 3.1.1.3. \( s_y \leq l \) and \( t_y > l \) (\( t_y \leq l \) and \( s_y > l \)). \((L(m, n, k, l), s, t)\) is in Subcase 1.4 of Lemma 4.4.

Subcase 3.1.2. \( m - k > 2 \) and \( n - l = 2 \). \((L(m, n, k, l), s, t)\) is in Case 2 of Lemma 4.4.

Subcase 3.1.3. \( m - k > 2 \), \( n - l > 2 \), and \( k > 1 \).

Subcase 3.1.3.1. \( s_y, t_y \leq m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 3.1 of Lemma 4.4.

Subcase 3.1.3.2. \( s_y, t_y > m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 3.2 or 3.3 of Lemma 4.4.

Subcase 3.1.3.3. \( s_y \leq m - k \) and \( t_y > m - k \). \((L(m, n, k, l), s, t)\) is in Subcase 3.4 of Lemma 4.4.

Subcase 3.1.4. \( m - k > 2 \), \( n - l > 2 \), and \( k = 1 \).

Subcase 3.1.4.1. \( l > 1 \). \((L(m, n, k, l), s, t)\) is in Case 4 of Lemma 4.4.

Subcase 3.1.4.2. \( l = 1 \).

Subcase 3.1.4.2.1. \( n = m = 5 \).

Subcase 3.1.4.2.1.1. \( s_y, t_y \leq 3 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.1.1, 5.1.3, or 5.2 of Lemma 4.4.

Subcase 3.1.4.2.1.2. \( s_y, t_y > 3 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.1.1, 5.1.2, or 5.2 of Lemma 4.4.

Subcase 3.1.4.2.1.3. \( s_y \leq 3 \) and \( t_y > 3 \) (\( t_y \leq 3 \) and \( s_y > 3 \)). \((L(m, n, k, l), s, t)\) is in Subcase 5.1.1, 5.1.4, or 5.2 of Lemma 4.4.

Subcase 3.1.4.2.2. \( m > 5 \) and \( n = 5 \).

Subcase 3.1.4.2.2.1. \( s_y, t_y \leq m - 3 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.3.1.1 of Lemma 4.4.

Subcase 3.1.4.2.2.2. \( s_y, t_y > m - 3 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.3.1.2 of Lemma 4.4.

Subcase 3.1.4.2.2.3. \( s_y \leq m - 3 \) and \( t_y > m - 3 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.3.1.3 of Lemma 4.4.

Subcase 3.1.4.2.3. \( m = 5 \) and \( n > 5 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.3.2 of Lemma 4.4.

Subcase 3.1.4.2.4. \( m, n > 5 \). \((L(m, n, k, l), s, t)\) is in Subcase 5.3.3 of Lemma 4.4.

Subcase 3.2. \( m - k = \text{odd} \) and \( n - l = \text{odd} \).

Subcase 3.2.1. \( s_x, t_x \leq l \). \((L(m, n, k, l), s, t)\) is in Case 1 of Lemma 4.5.

Subcase 3.2.2. \( s_x, t_x > l \). \((L(m, n, k, l), s, t)\) is in Case 2 of Lemma 4.5.

Subcase 3.2.3. \( s_y \leq l \) and \( t_y > l \) (\( t_y \leq l \) and \( s_y > l \)). \((L(m, n, k, l), s, t)\) is in Case 3 of Lemma 4.5.

All possible cases are exhausted, and the proof of Theorem 4.1 is completed.

By Theorem 3.2 and Lemmas 4.2–4.5, we have the following result:

**Theorem 4.6.** \((L(m, n, k, l))\) has a Hamiltonian \((s, t)\)-path if and only if \((L(m, n, k, l), s, t)\) is acceptable.

Theorem 4.7 summarizes our results.

**Theorem 4.7.** In an acceptable \( P(L(m, n, k, l), s, t)\), a Hamiltonian \((s, t)\)-path can be found in linear time.

**Proof.** The algorithm initially divides \( L(m, n, k, l) \) into some rectangular grid subgraphs in \( O(1) \) by Lemmas 4.2–4.5. Then builds Hamiltonian cycles or paths in these grid subgraphs in linear time. Next, it combines Hamiltonian cycles and paths for constructing a Hamiltonian \((s, t)\)-path in \( O(1) \). Therefore, the algorithm has linear-time complexity.

5. Conclusion and future work

We gave necessary and sufficient conditions for the existence of a Hamiltonian path in \( C \)-shaped grid graphs between two given vertices, which are a special type of solid grid graphs. The Hamiltonian path problem is \( NP \)-complete in general grid graphs [12], it remains open if the problem is polynomially solvable in solid grid graphs. Further study can be done on the Hamiltonian path problem in other special classes of graphs, in order to find way to solve the problem for solid grid graphs.
References


